ISRAEL JOURNAL OF MATHEMATICS **TBD** (2023), 1–49
DOI: 10.1007/s11856-023-2477-9

ON THE STRUCTURE OF THE NEHARI SET ASSOCIATED TO A SCHRÖDINGER–POISSON SYSTEM WITH PRESCRIBED MASS: OLD AND NEW RESULTS

BY

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ABSTRACT

In this paper we apply the fibering method of Pohozaev and the notion of extremal values introduced by Il'yasov to a Schrödinger–Poisson system, with prescribed L^2 norm of the unknown, in the whole $\mathbb{R}^3$. The method makes clear the role played by the special exponents $p = 3$, $p = 8/3$, $p = 10/3$.

In addition to showing that old results can be obtained in a unified way, we exhibit also new ones.

* Gaetano Siciliano was partially supported by Fapesp grant 2019/27491-0, CNPq grant 304660/2018-3 and Capes.

** Kaye Silva was partially supported by CNPq/Brazil under Grant 408604/2018-2.
Received January 14, 2021 and in revised form September 21, 2021

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1. Introduction

It is well-known that the following Schrödinger equation (where all the physical constants are normalized to unity),

$$i\partial_t \psi = -\Delta_x \psi + q(|\cdot|^{-1} * |\psi|^2)\psi - \lambda|\psi|^{8/3}\psi, \quad \psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}, \tag{1.1}$$

has a relevant role in many physical models. Here i is the imaginary unit, Δ_x is the Laplacian with respect to the spatial variables, $*$ is the x-convolution and q, λ are positive parameters.

As we can see, two types of potentials, different in nature, appear in the equation: the first one is the Hartree (or Coulomb) potential given by

$$V_{\mathrm{H}}(\cdot,t) = |\cdot|^{-1} * |\psi(\cdot,t)|^2$$

which is nonlocal and the second one is the Slater approximation of the exchange term, given by $|\psi|^{8/3}\psi$, which is local, although nonlinear. In this context q and λ are also called, respectively, the **Poisson constant** which represents the electric charge, and the **Slater constant**. The nonlocal potential can be seen as "generated" by the same wave function ψ, in virtue of the Poisson equation

$$-\Delta_x V_{\mathrm{H}} = 4\pi|\psi|^2.$$

A particular feature of (1.1) is that, due to the invariance by $U(1)$ gauge-transformations and the invariance by time translations, by the Noether theorem, on the solutions ψ the quantities

$$M(\psi)(t) = \int |\psi(x,t)|^2 dx$$

and

$$\begin{aligned} E(\psi)(t) =& \frac{1}{2}\int |\nabla_x \psi(x,t)|^2 dx + \frac{q}{4}\int \Big(\frac{1}{|\cdot|} * |\psi(\cdot,t)|^2\Big)|\psi(x,t)|^2 dx \\ & - \frac{3\lambda}{8}\int |\psi(x,t)|^{8/3} dx \end{aligned}$$

are conserved in time. In physical terms they are called respectively **mass** and **energy** of the solution.

Since the Hartree potential and the Slater term have different signs in the energy functional, they are in competition and then a different behaviour of E is expected depending on the values of the parameters q and λ. For more physical details and the derivation of (1.1) see, e.g., the seminal works [6, 15, 18, 19, 25] and the references therein. We mention that the above equation has been derived also in the framework of Abelian Gauge Theories in [5] and called the **Schrödinger–Poisson system**.

In this work we consider the problem of finding standing waves solutions

$$\psi(x,t) = u(x)e^{-i\ell t} \quad u : \mathbb{R}^3 \to \mathbb{R}, \quad \ell \in \mathbb{R}$$

to the above equation (1.1) under the mass constraint (as justified by the mass conservation law) and where the exponent $8/3$ is replaced by a more general p.

More specifically, we consider the problem of finding $\ell \in \mathbb{R}$ and $u \in H^1(\mathbb{R}^3)$ satisfying

$$\begin{cases} -\Delta u + q\phi_u u - \lambda |u|^{p-2}u = \ell u, & \text{in } \mathbb{R}^3 \\ \int u^2 = r \end{cases} \tag{1.2}$$

(from now on all the integrals will be in $\mathbb{R}^3$ and dx will be omitted) where

- $q, \lambda, r > 0$ are given parameters,
- $p \in (2, 6)$,
- $\phi_u \in D^{1,2}(\mathbb{R}^3)$ is the unique solution of the Poisson equation $-\Delta\phi = 4\pi u^2$ in $\mathbb{R}^3$, that can be represented, for $u \in H^1(\mathbb{R}^3)$, as
$$\phi_u = \frac{1}{|\cdot|} * u^2.$$

In particular we are interested in finding **ground state solutions** u, namely the solutions with minimal energy in the sense specified below.

The usual way to attack the problem is by variational methods. Indeed the weak solutions of equation (1.2) are easily seen to be critical points of the energy functional

$$E(u) := E_{q,\lambda}(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{q}{4}\int \phi_u u^2 - \frac{\lambda}{p}\int |u|^p,$$

constrained to the $L^2(\mathbb{R}^3)$ sphere

$$S_r = \left\{ u \in H^1(\mathbb{R}^3) : \int u^2 = r \right\},$$

as it follows by the Lagrange multiplier rule; in this case $\ell \in \mathbb{R}$ is the Lagrange multiplier associated to the critical point. Then this problem fits into the question of finding critical points of the energy on the mass constraint (see [4]).

An interesting problem is the search for ground states solutions, namely the minima of E on S_r, since they give rise to stable standing waves solutions for the evolution problem (1.1). The problem is not trivial since the behaviour of E depends on q, λ, p but actually also the value r has a main role.

The search of minima for similar problems has been addressed by Lions in the celebrated paper [16] where he studied the problem from a mathematical point of view and established, roughly speaking, that the existence of minimizers is equivalent to the strict sub-additive inequality

$$\inf_{S_r} E < \inf_{S_s} E + \inf_{S_{r-s}} E, \quad 0 < s < r. \tag{1.3}$$

In turn, this is equivalent to showing that dichotomy does not occur when one tries to apply the concentration-compactness principle of Lions (since the vanishing is avoided due to $\inf_{S_r} E < 0$). As showed in recent papers, inequality (1.3) does hold in certain intervals that depend on the values of p. See, e.g., Bellazzini and Siciliano [2, 3], Catto and Lions [8], Sánchez and Soler [22], Jeanjean and Luo [12], Catto et al. [7] and Colin and Watanabe [9].

We point out that in the last decades equations like (1.2), even without the mass constraint, have been extensively studied due to the mathematical challenges raised by the nonlocal term ϕ_u and its competition with the local nonlinearity.

The aim of this paper is to establish, by using the fibration method of Pohozaev developed in [20] and the notion of extremal values introduced in Il'yasov [10], a general framework which permits us to search for global minimizers of E over components of a suitable Nehari type set. These components are shown to be differentiable manifolds and natural constraints for the energy functional. The method proposed in this work makes more clear the relation between minimizers of E restricted to S_r and the parameters q, λ, p and r. In particular, it sheds some light on the role of special exponents p appearing in the Schrödinger–Poisson system: $p = 8/3, p = 3, p = 10/3$. Moreover, it relates the strict sub-additive inequality with topological properties of some natural curves that cross the Nehari manifolds as r varies.

Indeed beside recovering known results, we get also new ones and interesting estimates.

We think that this fibering approach can be used to solve also other different problems involving suitable constraints (different from the L^2-norm).

To conclude this Section, we point out that recently the fibration method together with the notion of extremal values, that guarantees regions of parameters in which the Nehari manifold method can be applied to prove existence of solutions, has been used successfully also in other types of equations as in [11,23,24].

2. Statement of the results

In this paper we obtain five types of results, all based on the introduction of a suitable Nehari set type for the functional E restricted to S_r and on its properties. Before we state our results, we need to introduce some notation.

First of all, by using standard methods, it is easy to see (see Proposition 4.2) that every critical point of E restricted to S_r belongs to the **Nehari type set**

$$\mathcal{N}_r := \mathcal{N}_{r,q,\lambda} = \Big\{u \in S_r : \int |\nabla u|^2 + \frac{q}{4}\int \phi_u u^2 - \frac{3(p-2)}{2p}\lambda \int |u|^p = 0\Big\}.$$

Indeed this set has been already introduced in [1, 12]. However, by means of the fibering method, we are able to decompose $\mathcal{N}_r$ into the subsets

$$\mathcal{N}_r^+ = \Big\{u \in \mathcal{N}_r : \int |\nabla u|^2 - \frac{3(p-2)(3p-8)}{4p}\lambda \int |u|^p > 0\Big\},$$
$$\mathcal{N}_r^0 = \Big\{u \in \mathcal{N}_r : \int |\nabla u|^2 - \frac{3(p-2)(3p-8)}{4p}\lambda \int |u|^p = 0\Big\},$$
$$\mathcal{N}_r^- = \Big\{u \in \mathcal{N}_r : \int |\nabla u|^2 - \frac{3(p-2)(3p-8)}{4p}\lambda \int |u|^p < 0\Big\},$$

so that $\mathcal{N}_r = \mathcal{N}_r^+ \cup \mathcal{N}_r^0 \cup \mathcal{N}_r^-$ and even more, we show that whenever nonempty, $\mathcal{N}_r^+$ and $\mathcal{N}_r^-$ are differentiable manifolds of codimension 2 in $H^1(\mathbb{R}^3)$ and a natural constraint for the functional E restricted to S_r (see Theorem 4.9).

One of the main ingredients in our proofs will be the analysis of the functional

$$R_p(u) = \frac{\left(\int |\nabla u|^2\right)^{\frac{3p-8}{4(p-3)}} \left(q\int \phi_u u^2\right)^{\frac{10-3p}{4(p-3)}}}{\left(\lambda \int |u|^p\right)^{\frac{1}{2(p-3)}}}, \quad u \in S_1,\ p \neq 3.$$

This functional is obtained with the help of Pohozaev's fibration method and is the so-called nonlinear Rayleigh quotient introduced by Il'yasov in [10]. Its topological properties are related with existence and non-existence of solutions for our problem and, although not in this form, this functional was already used in [7]. See also [9] where the nonlinear Rayleigh quotient was found by fixing $r > 0$ and considering q as a parameter; however, this is different from our approach, the main goal of which is to analyse the topological properties of the Nehari set also when r varies.

A rough summary of the results proved here is the following:

(I) we present new inequalities involving functions in $H^1(\mathbb{R}^3)$ and its Newtonian potential;

(II) we show the structure of the Nehari set $\mathcal{N}_r$ and related existence/non-existence results;

(III) we estimate the smallness of $r > 0$ which guarantees the existence of a minimum stated in [3, Theorem 4.1];

(IV) we prove existence of global minimizers at a positive energy level;

(V) we estimate the smallness of $r>0$ which permits to apply the methods used in [1] and show the existence of a solution.

Let us better detail our results.

(I) Our first results concern new inequalities we were not able to find in the literature. They are obtained by exploring the functional R_p.

THEOREM 2.1: *For each $p\in[10/3,6)$ there exists a constant $C_{q,\lambda,p}>0$ such that*

$$\lambda\int|u|^p\le C_{q,\lambda,p}\frac{(\int|\nabla u|^2)^{\frac{3p-8}{2}}(\int u^2)^{2(p-3)}}{(q\int\phi_u u^2)^{\frac{3p-10}{2}}},\quad \forall u\in H^1(\mathbb{R}^3)\setminus\{0\}.$$

We remark that similar inequalities are known in the literature, see Catto et al. [7] for the case $p\in[8/3,10/3]$.

We have also the following inequality whose proof will be more involved than the previous theorem.

THEOREM 2.2: *For each $p\in(2,3)$ there exists a constant $C_{q,\lambda,p}>0$ such that*

$$\lambda\int|u|^p\ge C_{q,\lambda,p}\frac{(\int|\nabla u|^2)^{\frac{3p-8}{2}}(\int u^2)^{2(p-3)}}{(q\int\phi_u u^2)^{\frac{3p-10}{2}}},\quad \forall u\in H^1(\mathbb{R}^3)\setminus\{0\}.$$

(II) The second type of results deal with the structure of $\mathcal{N}_r$ and its consequences. The situation will be different in the cases $p\neq3$ and $p=3$ and related existence/non-existence results are obtained.

THE CASE $p\neq3$. For each $p\in(2,6)\setminus\{3\}$, define the infimum of E over a subset of the Nehari set, by

$$I_r:=I_{r,q,\lambda}=\inf\{E(u):u\in\mathcal{N}_r^+\cup\mathcal{N}_r^0\}.$$

In particular $\inf_{S_r}E\le I_r$. With our approach we are able to show the following.

THEOREM 2.3: *The following hold:*

(i) *If $p\in(2,8/3)$, then $\mathcal{N}_r=\mathcal{N}_r^+\neq\emptyset$ and $-\infty<I_r=\inf_{S_r}E<0$ for all $r>0$.*

(ii) *If $p\in(10/3,6)$, then $\mathcal{N}_r=\mathcal{N}_r^-\neq\emptyset$ and $I_r=-\infty$ for all $r>0$.*

(iii) *If $p=8/3$, then $\mathcal{N}_r=\mathcal{N}_r^+\neq\emptyset$ and $-\infty<I_r=\inf_{S_r}E<0$ for all $r>0$.*

(iv) *If $p=10/3$, then there exists a constant $K_{\mathrm{GN}}>0$ such that $\mathcal{N}_r\neq\emptyset$ if, and only if,*

$$\frac{5}{3K_{\mathrm{GN}}}\frac{1}{\lambda}<r^{2/3}.$$

In case $\mathcal{N}_r\neq\emptyset$ we have that $\mathcal{N}_r=\mathcal{N}_r^-$ and $I_r=-\infty$.

(v) *If* $p \in (8/3, 3)$, *then* $\mathcal{N}_r^+, \mathcal{N}_r^0, \mathcal{N}_r^-$ *are non-empty and* $I_r < 0$ *for all* $r > 0$.

(vi) *If* $p \in (3, 10/3)$, *then there exist* $0 < r^* < r_0^*$ *such that*

(1) $\mathcal{N}_r^+, \mathcal{N}_r^-$ *are non-empty if, and only if,* $r > r^*$.

(2) $\mathcal{N}_r^0 \neq \emptyset$ *if, and only if,* $r \geq r^*$.

Moreover, if $r > r_0^*$, *then* $I_r = \inf_{S_r} E < 0$ *while if* $r \in [r^*, r_0^*]$, *then* $I_r \geq 0$.

In the above Theorem we presented the statements in that order due to the techniques used in the proofs, which are similar, respectively, for (i)–(ii), (iii)–(iv) and (v)–(vi).

Theorem 2.3 (parts of it) can be found in most of the works cited until now, in particular we would like to refer the reader to the works [7, 12], where some calculations can be found explicitly.

Our contribution here is to show how, with a general framework, it is possible to connect all these results with the partitioning of the Nehari set $\mathcal{N}_r$ in terms of $\mathcal{N}_r^+, \mathcal{N}_r^0, \mathcal{N}_r^-$. Moreover we give a characterisation of the quantities r^*, r_0^* in terms of R_p, namely

$$r^* = \Big(4\frac{10-3p}{3p-8}\Big)^{\frac{3p-10}{4(p-3)}} \Big(\frac{4p}{3(p-2)(3p-8)}\Big)^{\frac{1}{2(p-3)}} \inf_{w\in S_1} R_p(w)$$

and

$$r_0^* = \Big(\frac{2(10-3p)}{3p-8}\Big)^{\frac{3p-10}{4(p-3)}} \Big(\frac{p}{3p-8}\Big)^{\frac{1}{2(p-3)}} \inf_{w\in S_1} R_p(w)$$

(see (5.2)) and it will be evident that they are related to some geometrical properties of the Nehari set (see Proposition 4.5). Note that (as already known) for $p \in [10/3, 6)$ we have $I_r = -\infty$, which is equivalent to $\mathcal{N}_r = \mathcal{N}_r^-$ and also suggests a mountain pass geometry (see Bellazini et al. [1]). Observe that items (iv) and (vi) of Theorem 2.3 give also results of the non-existence of critical points of E over S_r depending on r. Indeed since every critical point of E constrained to S_r belongs to $\mathcal{N}_r$, it follows that if $\mathcal{N}_r$ is empty, then there is no critical point at all; therefore as an immediate consequence of Theorem 2.3 we infer

Corollary 1: *The functional* E *constrained to* S_r *has no critical points if:*

(i) $p = 10/3$ *and* $\frac{5}{3K_{\rm GN}}\frac{1}{\lambda} \geq r^{2/3}$,

(ii) $p \in (3, 10/3)$ *and* $r < r^*$.

Note that the results in Theorem 2.3 and Corollary 1 are independent of $q > 0$ and the unique case in which λ has a role is when $p = 10/3$.

THE CASE $p = 3$. Here the situation changes drastically in the sense that r no longer has a role and the properties of the Nehari sets depend on q and λ. In order to make clear this dependence, we use the notation $\mathcal{N}_{q,\lambda}, I_{q,\lambda}, \ldots$ instead of the previous $\mathcal{N}_r, I_r \ldots$.

We prove the following:

THEOREM 2.4: *Let $p = 3$ and $r > 0$. For each fixed $q > 0$, there exist positive constants $\lambda_q^* < \lambda_{0,q}^*$ such that*

(i) *$\mathcal{N}_{q,\lambda}^+, \mathcal{N}_{q,\lambda}^-$ are non-empty if, and only if $\lambda > \lambda_q^*$. Moreover, if $\lambda > \lambda_q^*$ then $\mathcal{N}_{q,\lambda}^0 \neq \emptyset$.*

(ii) *If $\lambda > \lambda_{0,q}^*$, then $I_{q,\lambda} = \inf_{S_r} E < 0$, while if $\lambda \in (0, \lambda_{0,q}^*)$, then $I_{q,\lambda} \geq 0$.*

Similarly to the quantities r_0^*, r^*, the quantities $\lambda_{0,q}^*, \lambda_q^*$ have a geometrical interpretation and are given by

$$\lambda_{0,q}^* = \left(\frac{9}{2}\right)^{1/2} q^{1/2} \inf_{w \in S_1} \frac{(\int |\nabla w|^2 \int \phi_w w^2)^{1/2}}{\int |w|^3}$$

and

$$\lambda_q^* = 2q^{1/2} \inf_{w \in S_1} \frac{(\int |\nabla w|^2 \int \phi_w w^2)^{1/2}}{\int |w|^3}.$$

We observe that, unlike Theorem 2.3 item (vi), in Theorem 2.4 we were not able to describe the behavior of $\mathcal{N}_{q,\lambda_{0,q}^*}$. This is due to the fact that $u \in \mathcal{N}_{q,\lambda_{0,q}^*}$ if, and only if, u is a minimizer of the quotient

$$\frac{(\int |\nabla w|^2 \int \phi_w w^2)^{1/2}}{\int |w|^3} \quad \text{on } S_1.$$

The fact that the above quotient is bounded away from zero is due to Lions [17], however, it is an open problem if this functional has a minimizer. As was pointed out in [7], the minimizers of that functional also are (up to some constant) global minimizers of $I_{q,\lambda_{0,q}^*}$ and $I_{q,\lambda_{0,q}^*} = 0$. As before, we can deduce by using Theorem 2.4 a non-existence result.

COROLLARY 2: *Let $p = 3$ and $r, q > 0$. The functional E constrained to S_r has no critical points if $\lambda \in (0, \lambda_q^*)$.*

(III) Our third type of result complements [3, Theorem 4.1]. Indeed in [3] (see also [7]) the authors proved, among other things, that for small r, there exist minimizers for E over S_r. With our approach we are able to give a quantitative estimate on the "smallness" of r in terms of R_p which guarantees the existence of minimizers.

Theorem 2.5: *Let $p \in (2,3)$. Then for each*

$$r \in \Big(0, \Big[\frac{1}{2}\Big(\frac{2(p-2)}{p}\Big)^{\frac{2}{3p-8}}\Big]^{\frac{3p-8}{4(3-p)}} \inf_{w \in S_1} R_p(w)\Big)$$

there exists $u \in S_r$ satisfying $E(u) = \min_{S_r} E$.

(IV) The fourth type of result deals with the existence of a global minimizer with positive energy when $p \in (3, 10/3)$ which has never been treated in the literature. In this case the inequality $\inf_{S_r} E < 0$ is no longer true for $r \in [r^*, r_0^*]$. Moreover, $\inf_{S_r} E = 0$ if $0 < r \le r_0^*$ and it is not achieved.

We extend these results by showing the existence of local minimizers for E on S_r with positive energy, when r belongs to a neighborhood of r_0^*. Unfortunately we are not able to cover the whole range $(3, 10/3)$. Our result is

Theorem 2.6: *There exists $p_0 \in (3, 10/3)$ such that if $p \in (p_0, 10/3)$, then*

- (i) *the function $[r^*, \infty) \ni r \mapsto I_r$ is decreasing, $I_{r_0^*} = 0$ and $I_r > 0$ for $r \in [r_*, r_0^*)$;*
- (ii) *for each $r \in [r^*, r_0^*)$ there exists $u \in \mathcal{N}_r^+ \cup \mathcal{N}_r^0$ such that $I_r = E(u)$;*
- (iii) *there exists $\varepsilon > 0$ such that if $r \in (r_0^* - \varepsilon, r_0^*)$, then there exists $u \in \mathcal{N}_r^+$ such that $I_r = E(u)$.*

Indeed we find explicitly the number

$$p_0 = \frac{73 + \sqrt{145}}{27}$$

which is new in the literature.

We believe it is an interesting problem to study the case $p \in (3, p_0]$.

We point out that similar results to our Theorem 2.6 have been obtained in [13, Theorem 1.1] for a different equation involving a quasilinear term. However, with our new approach we present a characterisation of the extremal value r_0^*; see (5.2).

(V) Finally we study the smallness of r which guarantees the existence of solutions by the methods used in [1]. With this aim let

$$J_r = \inf\{E(u) : u \in \mathcal{N}_r^-\}.$$

We have

THEOREM 2.7: *Let $p \in [10/3, 6)$. Then for each*

$$r \in \Big(0, \Big(\frac{2(6-p)}{5p-12}\Big)^{\frac{3p-10}{4(p-3)}} \Big(\frac{3p}{5p-12}\Big)^{\frac{1}{2(p-3)}} \inf_{u \in S_1} R_p(u)\Big)$$

there exists $u \in \mathcal{N}_r^-$ such that $J_r = E(u)$.

ORGANISATION OF THE PAPER. The paper is organized as follows.

In Section 3 we study deeply the Rayleigh quotient R_p. Indeed most of the results are based on its properties. We then give the proof of Theorem 2.1 and Theorem 2.2.

In Section 4 we introduce the set $\mathcal{N}_r$ and give a description via the fibering method of its subsets $\mathcal{N}_r^+, \mathcal{N}_r^-, \mathcal{N}_r^0$ on which we study the energy functional E. In particular, we show that $\mathcal{N}_r^+, \mathcal{N}_r^-$ are differentiable manifolds and natural constraints for E (see Theorem 4.9).

In Section 5 we study deeply these sets depending on the parameters q, λ, p, r. This analysis will allow us to prove our second type of results, namely Theorem 2.3 and Theorem 2.4.

In Section 6 we show the subadditivity condition for I_r that will serve as a prerequisite for the subsequent Section.

In Section 7 we prove Theorem 2.5 and Theorem 2.6.

Section 8 is devoted to the proof of Theorem 2.7.

In Appendix A we give a new estimate concerning I_{r_1} and I_{r_2} for $r_1 < r_2$ obtained by means of the fibering approach.

NOTATION. As a matter of notation, throughout the paper we denote by $\|\cdot\|_p$ the L^p-norm in $\mathbb{R}^3$. We use $o_n(1)$ to denote a vanishing sequence. Given a function u and $t > 0$, we set

$$u^t(x) = t^{\frac{3}{2}} u(tx).$$

Note that $\|u\|_2 = \|u^t\|_2$.

3. The nonlinear Rayleigh quotient R_p

Let us start with a simple and general result whose proof is straightforward, so it is omitted.

PROPOSITION 3.1: *Suppose that $b \neq 0$, $ce - bf \neq 0$, $(bd - ae)/(ce - bf) > 0$, $(af - cd)/(ce - bf) > 0$, $A, B, C > 0$ and $p \in (2, 6) \setminus \{3\}$. Then the system*

$$\begin{cases} atA + brB + cr^{\frac{p}{2}-1}t^{\frac{3p}{2}-4}C = 0 \\ dtA + erB + fr^{\frac{p}{2}-1}t^{\frac{3p}{2}-4}C = 0 \end{cases} \tag{3.1}$$

admits a unique solution $r, t > 0$. Moreover, explicitly we have

$$r = \Big(\frac{bd - ae}{ce - bf}\Big)^{\frac{1}{2(p-3)}} \Big(\frac{af - cd}{ce - bf}\Big)^{\frac{3p-10}{4(p-3)}} \frac{A^{\frac{3p-8}{4(p-3)}} B^{\frac{3p-10}{4(p-3)}}}{C^{\frac{1}{2(p-2)}}}.$$

Recall the next two results.

LEMMA 3.2 (Catto et al. [7]): *For each $p \in (2, 6)$ and $r > 0$, there exists a sequence of functions $\{u_n\} \subset S_r$ and positive constants C_1, C_2 and C_3, satisfying*

$$\int |u_n|^p = C_1, \quad \int |\nabla u_n|^2 = C_2 n^{\frac{2}{3}}, \quad \int \phi_{u_n} u_n^2 \leq \frac{C_3}{n^{\frac{2}{3}}}, \quad \forall n \in \mathbb{N}.$$

LEMMA 3.3 (Catto et al. [7]): *Assume that $p \in [8/3, 3]$, then there exists a constant $C > 0$ such that*

$$\int |u|^p \leq C\Big(\int u^2\Big)^{2(3-p)} \Big(\int \phi_u u^2\Big)^{\frac{p-2}{2}} \Big(\int |\nabla u|^2\Big)^{p-2}, \quad \forall u \in H^1(\mathbb{R}^3). \tag{3.2}$$

If $p \in [3, 10/3]$, then there exists a constant $C > 0$ such that

$$\int |u|^p \leq C\Big(\int u^2\Big)^{2(p-3)} \Big(\int \phi_u u^2\Big)^{\frac{10-3p}{2}} \Big(\int |\nabla u|^2\Big)^{\frac{3p-8}{2}}, \quad \forall u \in H^1(\mathbb{R}^3). \tag{3.3}$$

Let us define for $p \in (2, 6) \setminus \{3\}$ the quotient

$$R_p(u) = \frac{(\int |\nabla u|^2)^{\frac{3p-8}{4(p-3)}} (q \int \phi_u u^2)^{\frac{10-3p}{4(p-3)}}}{(\lambda \int |u|^p)^{\frac{1}{2(p-3)}}}, \quad u \in S_1. \tag{3.4}$$

Note that in particular

$$R_{8/3}(u) = \frac{(\lambda \int |u|^{8/3})^{3/2}}{(q \int \phi_u u^2)^{3/2}} \quad \text{and} \quad R_{10/3} = \frac{(\int |\nabla u|^2)^{3/2}}{(\lambda \int |u|^{10/3})^{3/2}}. \tag{3.5}$$

The next result is just Lemma 3.2 and the inequality (3.3) of Lemma 3.3 rewritten in terms of our functional R_p.

PROPOSITION 3.4: *The functional R_p defined above is continuous. Moreover:*

(i) *if $p \in (2,3)$, then the functional R_p is unbounded from above,*
(ii) *if $p \in (3, 10/3]$, then the functional R_p is bounded away from 0.*

Proof. The continuity is obvious. To prove (i), if $\{u_n\}$ is the sequence given in Lemma 3.2, since $p < 3$, it follows that $R_p(u_n) \geq Cn$ where C is some positive constant. The proof of (ii) is a direct consequence of (3.3) of Lemma 3.3. ∎

For future reference we consider the system

$$\begin{cases} rt^2 \int |\nabla u|^2 + \frac{r^2 t}{4} q \int \phi_u u^2 - \frac{3(p-2)}{2p} r^{p/2} t^{\frac{3(p-2)}{2}} \lambda \int |u|^p = 0, \\ \frac{q}{2} r^2 t \int \phi_u u^2 - \frac{p-2}{p} r^{p/2} t^{\frac{3(p-2)}{2}} \lambda \int |u|^p = 0, \end{cases} \tag{3.6}$$

where $u \in S_1$ and $r, t > 0$. From Proposition 3.1 and recalling R_p defined in (3.4) and (3.5), we have that if $p \in (2,6) \setminus \{3\}$, then the system has a unique solution $(\widetilde{r}(u), \widetilde{t}(u))$ which is given by:

- if $p \in (2,3)$ with $p \neq 8/3$:

$$\begin{aligned} \widetilde{r}(u) &= \Big[\frac{1}{2}\Big(\frac{2(p-2)}{p}\Big)^{\frac{2}{3p-8}}\Big]^{\frac{3p-8}{4(3-p)}} R_p(u), \\ \widetilde{t}(u) &= \Big(\frac{p}{2(p-2)} \widetilde{r}(u)^{\frac{4-p}{2}} \frac{q}{\lambda} \frac{\int \phi_u u^2}{\int |u|^p}\Big)^{\frac{2}{3p-8}}; \end{aligned} \tag{3.7}$$

- if $p = 8/3$:

$$\begin{aligned} \widetilde{r}(u) &= \frac{1}{2^{3/2}} R_{8/3}(u), \\ \widetilde{t}(u) &= \frac{1}{2^{5/2}} \frac{(\lambda \int |u|^p)^{3/2}}{(q \int \phi_u u^2)^{1/2} \int |\nabla u|^2}. \end{aligned} \tag{3.8}$$

Remark 1: Note that $\widetilde{r}(u)$ as a function of p is continuous in $p = 8/3$ since

$$\lim_{p \to 8/3} \Big[\frac{1}{2}\Big(\frac{2(p-2)}{p}\Big)^{\frac{2}{3p-8}}\Big]^{\frac{3p-8}{4(3-p)}} = \frac{1}{2^{3/2}}.$$

We recall the following Hardy–Littlewood–Sobolev inequality (see [14, Theorem 4.3]):

THEOREM 3.5: *Assume that $1 < a, b < \infty$ satisfy*

$$\frac{1}{a} + \frac{1}{b} = \frac{5}{3}.$$

Then there exists a constant $H_{a,b} > 0$ *such that*

$$\left| \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{f(x)g(y)}{|x-y|} dxdy \right| \leq H_{a,b} \|f\|_a \|g\|_b, \quad \forall f \in L^a(\mathbb{R}^3), g \in L^b(\mathbb{R}^3).$$

Then we can prove the following result.

PROPOSITION 3.6: *For each* $p \in [10/3, 6)$, *there exists a constant* $C_{q,\lambda,p} > 0$ *such that*

$$R_p(u) \geq C_{q,\lambda,p}, \quad \forall u \in S_1.$$

Proof. We have by definition

$$R_p(u) = \frac{(\int |\nabla u|^2)^{\frac{3p-8}{4(p-3)}}}{(\lambda \int |u|^p)^{\frac{1}{2(p-3)}} (q \int \phi_u u^2)^{\frac{3p-10}{4(p-3)}}}, \quad u \in S_1.$$

We can assume that $\|\nabla u\|_2 = 1$ and hence the conclusion is a simple consequence of Sobolev embeddings and the Hardy–Littlewood–Sobolev inequality. ∎

More involved is the proof of the next result.

PROPOSITION 3.7: *For each* $p \in (2, 3)$, *there exists a constant* $C_{q,\lambda,p} > 0$ *such that*

$$R_p(w) \geq C_{q,\lambda,p}, \quad \forall w \in S_1.$$

Proof. First note that

$$R_p(w^t) = R_p(w) \quad \text{and} \quad \int |\nabla w^t|^2 = t^2 \int |\nabla w|^2 \quad \forall w \in S_1, \ t > 0.$$

From this it follows that

$$\inf_{w \in S_1} R_p(w) = \inf \left\{ R_p(w) : w \in S_1, \ \int |\nabla w|^2 = 1 \right\}. \tag{3.9}$$

Indeed, for any $\varepsilon > 0$ there exists $u \in S_1$ such that $R_p(u) \leq \inf_{S_1} R_p + \varepsilon$. Then, if we consider u^{t_*}, where $t_* \int |\nabla u|^2 = 1$, we have that

$$u^{t_*} \in S^1 \ \text{ and } \ \int |\nabla u^{t_*}|^2 = 1.$$

Consequently

$$\inf \left\{ R_p(w) : w \in S_1, \ \int |\nabla w|^2 = 1 \right\} \leq R_p(u^{t_*}) = R_p(u) \leq \inf_{S_1} R_p + \varepsilon$$

and (3.9) follows. The approach to prove the theorem will be different according to the values of p.

Case 1: $p \in (8/3, 3)$.

Assume on the contrary that there exists a sequence $\{w_n\} \subset S_1$ such that

$$R_p(w_n) \to 0 \quad \text{as } n \to +\infty.$$

Let $\widetilde{r}(w_n)$ and $\widetilde{t}(w_n)$ be the solutions of system (3.6), see (3.7), and set for brevity $r_n = \widetilde{r}(w_n)$ and $u_n = r_n^{1/2} w_n^{\widetilde{t}(w_n)}$. It is easy to see that, for all $n \in \mathbb{N}$,

$$\begin{cases} \int |\nabla u_n|^2 + \frac{q}{4}\int \phi_{u_n} u_n^2 - \frac{3(p-2)}{2p}\lambda \int |u_n|^p = 0, \\ \frac{q}{2}\int \phi_{u_n} u_n^2 - \frac{p-2}{p}\lambda \int |u_n|^p = 0. \end{cases} \tag{3.10}$$

Now observe that (3.10) is the same as [3, equation (4.9)] and therefore

$$E(u_n) = \frac{3-p}{2-p}\int |\nabla u_n|^2 < 0, \quad \forall n \in \mathbb{N},$$

which implies that $I_{r_n} \le E(u_n) < 0$ and hence, since $I_{r_n} \to 0$ as $n \to +\infty$, we obtain that $E(u_n) \to 0$ as $n \to \infty$. The last convergence implies [3, formula (4.10)]. Therefore, by following the proof of [3, Theorem 4.1., step 5, case (e)], we reach a contradiction and hence R_p is bounded from below over the sphere S_1.

To treat the other cases of p, we observe that, since $p < 3$, the Lemma is proved once we show that $R_p^{2(p-3)}$ is bounded above if $\|w\|_2 = \|\nabla w\|_2 = 1$.

Case 2: $p \in (12/5, 8/3]$.

By choosing $a = p/2$ and $b = 3p/(5p-6)$ from Theorem 3.5 we obtain

$$\int \phi_w w^2 \le H_{a,b}\|w^2\|_{p/2}\|w^2\|_b = H_{a,b}\|w\|_p^2\|w\|_{2b}^2, \quad \forall w \in H^1(\mathbb{R}^3).$$

Since $2 < 2b < p$, from the interpolation inequality we have that

$$\|w\|_{2b} \le \|w\|_p^{\frac{2(3-p)}{3(p-2)}}\|w\|_2^{\frac{p}{3(p-2)}},$$

and hence

$$\int \phi_w w^2 \le H_{a,b}\|w\|_p^{\frac{2p}{3(p-2)}}\|w\|_2^{\frac{2p}{3(p-2)}}.$$

Consequently, for a suitable constant $C_{p,q} > 0$ depending only on p and q, we get

$$\begin{aligned} R_p(w)^{2(p-3)} &= \frac{(\int |\nabla w|^2)^{\frac{3p-8}{2}} (q \int \phi_w w^2)^{\frac{10-3p}{2}}}{\lambda \int |w|^p} \\ &\leq \frac{C_{q,p}}{\lambda} \frac{(\int |w|^p)^{\frac{10-3p}{3(p-2)}}}{\int |w|^p} \\ &= \frac{C_{q,p}}{\lambda} \left(\int |w|^p \right)^{\frac{2(8-3p)}{3(p-2)}} \\ &\leq 2\frac{C_{q,p}}{\lambda}, \qquad\qquad \forall w \in S_1, \int |\nabla w|^2 = 1. \end{aligned}$$

Case 3: $p \in (2, 12/5]$.

We choose $a = b = 6/5$ in Theorem 3.5 and use the interpolation inequality to conclude that

$$\int \phi_w w^2 \leq H_{6/5,6/5} \|w\|_{12/5}^4 \leq H_{6/5,6/5} \|w\|_p^{\frac{6p}{6-p}} \|w\|_6^{\frac{2(12-5p)}{6-p}}, \quad \forall w \in H^1(\mathbb{R}^3).$$

From the Sobolev inequality we obtain that, for a suitable constant $S > 0$, depending only on p, we have

$$\int \phi_w w^2 \leq H_{6/5,6/5} S \|w\|_p^{\frac{6p}{6-p}}, \qquad \forall w \in S_1, \int |\nabla w|^2 = 1.$$

Consequently, for a suitable constant $C_{p,q} > 0$ depending only on p and q, we have

$$\begin{aligned} R_p^{2(p-3)}(w) &= \frac{(\int |\nabla w|^2)^{\frac{3p-8}{2}} (q \int \phi_w w^2)^{\frac{10-3p}{2}}}{\lambda \int |w|^p} \\ &\leq \frac{C_{q,p}}{\lambda} \frac{(\int |w|^p)^{\frac{3(10-3p)}{6-p}}}{\int |w|^p} \\ &= \frac{C_{q,p}}{\lambda} \left(\int |w|^p \right)^{\frac{8(3-p)}{6-p}} \\ &\leq 2\frac{C_{q,p}}{\lambda}, \qquad\qquad \forall w \in S_1, \int |\nabla w|^2 = 1, \end{aligned}$$

and hence the proof is concluded. ∎

As a consequence of the previous proposition, we have the new inequalities stated in Theorem 2.1 and Theorem 2.2.

3.1. Proof of Theorem 2.1 and Theorem 2.2. They follows respectively by Proposition 3.6 and Proposition 3.7 with a simple L^2-normalization.

4. Natural constraints for E

In this Section we prove the existence of a natural constraint for the energy functional E restricted to S_r. Although such a constraint appeared already in [1, 12], the proof that it is a manifold and a natural constraint seems to be new.

We start with a well-known Pohozaev identity, which will be quite useful: for $a, b, c, d \in \mathbb{R}$ consider the equation

$$\begin{cases} -a\Delta u + bu + c\phi_u u + d|u|^{p-2}u = 0, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{4.1}$$

and define $P : H^1(\mathbb{R}^3) \to \mathbb{R}$ by

$$P(u) = \frac{a}{2}\int |\nabla u|^2 + \frac{3b}{2}\int u^2 + \frac{5c}{4}\int \phi_u u^2 + \frac{3d}{p}\int |u|^p.$$

Then we have the following Pohozaev identity; see [21, Theorem 2.2]:

Proposition 4.1: *If u satisfies* (4.1), *then* $P(u) = 0$.

As a consequence we get the next result which is already known (see [12, Lemma 2.1]), however, we prove it for completeness.

Proposition 4.2: *Assume that $u \in S_r$ is a critical point of E restricted to S_r. Then*

$$\int |\nabla u|^2 + \frac{q}{4}\int \phi_u u^2 - \frac{3(p-2)}{2p}\lambda \int |u|^p = 0.$$

Proof. Indeed, from the Lagrange multiplier rule there exists $\mu \in \mathbb{R}$ such that $E'(u) = \mu u$, that is, u is a solution of

$$-\Delta u + q\phi_u u - \lambda |u|^{p-2}u = \mu u.$$

In particular, u satisfies

$$\int |\nabla u|^2 + q\int \phi_u u^2 - \lambda \int |u|^p = \mu \int u^2$$

and by Proposition 4.1 also

$$\frac{1}{2}\int |\nabla u|^2 - \frac{3}{2}\mu \int u^2 + \frac{5}{4}q\int \phi_u u^2 - \frac{3\lambda}{p}\int |u|^p = 0$$

which together give the desired equality. ∎

Proposition 4.2 justifies the introduction of the set

$$\mathcal{N}_{r,q,\lambda} := \left\{ u \in S_r : \int |\nabla u|^2 + \frac{q}{4} \int \phi_u u^2 - \frac{3(p-2)}{2p} \lambda \int |u|^p = 0 \right\}, \tag{4.2}$$

since it contains any solution u of (1.2). In the following we will simply write $\mathcal{N}_r$. Define also

$$\mathcal{N}_r^+ = \left\{ u \in \mathcal{N}_r : \int |\nabla u|^2 - \frac{3(p-2)(3p-8)}{4p} \lambda \int |u|^p > 0 \right\}, \tag{4.3}$$

$$\mathcal{N}_r^0 = \left\{ u \in \mathcal{N}_r : \int |\nabla u|^2 - \frac{3(p-2)(3p-8)}{4p} \lambda \int |u|^p = 0 \right\}, \tag{4.4}$$

$$\mathcal{N}_r^- = \left\{ u \in \mathcal{N}_r : \int |\nabla u|^2 - \frac{3(p-2)(3p-8)}{4p} \lambda \int |u|^p < 0 \right\}. \tag{4.5}$$

Just in Subsection 5.2 it will be more convenient to express explicitly the dependence on q and λ, instead of r, since they will have an important role.

To obtain basic estimates for the elements of $\mathcal{N}_r$ let us recall the Gagliardo–Nirenberg inequality:

$$\int |u|^p \leq K_{\mathrm{GN}} \left(\int |\nabla u|^2 \right)^{\frac{3(p-2)}{4}} \left(\int u^2 \right)^{\frac{6-p}{4}}, \quad \forall u \in H^1(\mathbb{R}^3), \tag{4.6}$$

where $K_{\mathrm{GN}} > 0$, hereafter, is the Gagliardo–Nirenberg constant which depends only on p. Then we have

Proposition 4.3: *Let $r, \lambda > 0$ and $u \in \mathcal{N}_r$.*

(1) *For $p \in (2, 10/3)$, we have*

$$\int |\nabla u|^2 \leq K_{\mathrm{GN}}^{\frac{4}{(10-3p)}} \left(\frac{3(p-2)\lambda}{2p} \right)^{\frac{4}{10-3p}} r^{\frac{6-p}{10-3p}}.$$

(2) *For $p = 10/3$, we have*

$$\frac{5}{3\lambda K_{\mathrm{GN}}} \leq r^{2/3}.$$

(3) *If $p \in (3, 10/3)$, then there exist constants $c_p, c_p' > 0$ such that*

$$\int |\nabla u|^2 \geq \frac{c_p}{r} \quad \textit{and} \quad \int |u|^p \geq \frac{c_p'}{\lambda r}.$$

(4) *For $p \in (10/3, 6)$, we have*

$$\int |\nabla u|^2 \geq K_{\mathrm{GN}}^{\frac{4}{(10-3p)}} \left(\frac{3(p-2)\lambda}{2p} \right)^{\frac{4}{10-3p}} r^{\frac{6-p}{10-3p}}.$$

Proof. First observe that for any $u \in \mathcal{N}_r$ we have

$$\int |\nabla u|^2 \leq \frac{3(p-2)}{2p}\lambda \int |u|^p. \tag{4.7}$$

Combining (4.7) with the Gagliardo–Nirenberg inequality (4.6) we infer, for any $u \in \mathcal{N}_r$, that

$$\int |\nabla u|^2 \leq K_{\mathrm{GN}} \frac{3(p-2)\lambda}{2p} \Big(\int |\nabla u|^2 \Big)^{\frac{3(p-2)}{4}} r^{\frac{6-p}{4}}.$$

From this we deduce (1), (2) and (4).

Now assume that $p \in (3, 10/3)$. From [12, Lemma 2.3], there exist $c, c_p > 0$ positive constants, such that

$$\int |\nabla u|^2 + \frac{q}{4} \int \phi_u u^2 - \frac{3(p-2)}{2p}\lambda \int |u|^p \geq c \int |\nabla u|^2 - c_p \Big(\int |\nabla u|^2 \Big)^{\frac{3}{2}} r^{\frac{1}{2}}, \quad \forall u \in S_r.$$

Therefore

$$c \int |\nabla u|^2 - c_p \Big(\int |\nabla u|^2 \Big)^{\frac{3}{2}} r^{\frac{1}{2}} \leq 0, \quad \forall u \in \mathcal{N}_r,$$

and hence

$$\int |\nabla u|^2 \geq \Big(\frac{c}{c_p} \Big)^2 \frac{1}{r}, \quad \forall u \in \mathcal{N}_r \tag{4.8}$$

which is the first estimate in (3). The second one follows by (4.7) and (4.8). ∎

For the sake of completeness we observe, by looking at the proof of [12, Lemma 2.2 and Lemma 2.3], that the constants appearing in (3) of Proposition 4.3 are given explicitly by

$$\begin{gathered} c_p = \frac{p-3}{4-p} K_{\mathrm{GN}} \Big(\frac{3(p-2)(4-p)2^{7-p}}{p} \Big)^{1/(p-3)}, \quad c_p' = \frac{2p}{3(p-2)} \Big(\frac{c}{c_p} \Big)^2, \\ c = \frac{64\pi - 1}{64\pi} \end{gathered} \tag{4.9}$$

and do not depend on q.

As we will see, item (2) in Proposition 4.3 will be improved in Theorem 5.3.

4.1. The fibration for $\mathcal{N}_r$. We will use the fibration method of Pohozaev to study $\mathcal{N}_r$. Given $u \in S_1$, define the fiber map

$$\varphi_{r,q,\lambda,u} : t \in (0,\infty) \longmapsto E(r^{1/2}u^t) \in \mathbb{R}$$

where $u^t(x) = t^{\frac{3}{2}}u(tx) \in S_1$ and then $r^{1/2}u^t \in S_r$. Also in this case, until Subsection 5.2 we will write simply $\varphi_{r,u}$. Then explicitly we have

$$\varphi_{r,u}(t) = \frac{t^2}{2}r\int|\nabla u|^2 + \frac{t}{4}r^2q\int\phi_u u^2 - \frac{t^{3/2p-3}}{p}r^{p/2}\lambda\int|u|^p.$$

A simple computation gives the next

Lemma 4.4: *The fiber map $\varphi_{r,u}$ is a smooth function and*

$$\varphi'_{r,u}(t) = tr\int|\nabla u|^2 + \frac{r^2}{4}q\int\phi_u u^2 - \frac{3(p-2)}{2p}t^{\frac{3p}{2}-4}r^{p/2}\lambda\int|u|^p,$$

$$\varphi''_{r,u}(t) = r\int|\nabla u|^2 - \frac{3(p-2)(3p-8)}{4p}t^{\frac{3p}{2}-5}r^{p/2}\lambda\int|u|^p.$$

Then we can give a complete description of the fiber $\varphi_{r,u}$.

Proposition 4.5: *For each $u \in S_1$ the following statements hold:*

(I) *If $p \in (2, 8/3)$, then $\varphi_{r,u}$ has only one critical point at $t_r^+(u)$ which is a global minimum with $\varphi''_{r,u}(t_r^+(u)) > 0$.*

(II) *If $p = 8/3$, we have:*

(1) *if*

$$\frac{r^2}{4}\int\phi_u u^2 - \frac{r^{p/2}}{p}\lambda\int|u|^p < 0,$$

then $\varphi_{r,u}$ has only one critical point at $t_r^+(u)$ which is a global minimum with $\varphi''_{r,u}(t_r^+(u)) > 0$;

(2) *if*

$$\frac{r^2}{4}\int\phi_u u^2 - \frac{r^{p/2}}{p}\lambda\int|u|^p \geq 0,$$

then $\varphi_{r,u}$ is strictly increasing and has no critical points.

(III) *If $p \in (8/3, 10/3)$, then there are three possibilities:*

(1) *$\varphi_{r,u}$ has exactly two critical points at $t_r^-(u) < t_r^+(u)$. Moreover, $t_r^+(u)$ corresponds to a local minimum while $t_r^-(u)$ corresponds to a local maximum with $\varphi''_{r,u}(t_r^+(u)) > 0$ and $\varphi''_{r,u}(t_r^-(u)) < 0$;*

(2) *$\varphi_{r,u}$ is strictly increasing and has exactly one critical point at $t_r^0(u)$. Moreover, $t_r^0(u)$ corresponds to an inflection point;*

(3) *$\varphi_{r,u}$ is strictly increasing and has no critical points.*

(IV) *If $p = 10/3$, we have:*

(1) *if*

$$\frac{r}{2}\int |\nabla u|^2 - \frac{r^{p/2}}{p}\lambda \int |u|^p < 0,$$

then $\varphi_{r,u}$ has only one critical point at $t_r^-(u)$ which is a global maximum with $\varphi''_{r,u}(t_r^-(u)) < 0$;

(2) *if*

$$\frac{r}{2}\int |\nabla u|^2 - \frac{r^{p/2}}{p}\lambda \int |u|^p \geq 0,$$

then $\varphi_{r,u}$ is strictly increasing and has no critical points.

(V) *If $p \in (10/3, 6)$, then $\varphi_{r,u}$ has only one critical point at $t_r^-(u)$ which is a global maximum with $\varphi''_{r,u}(t_r^-(u)) < 0$.*

Proof. It is straightforward. ∎

A direct application of the Implicit Function Theorem shows that

Lemma 4.6: *Fix $u \in S_1$ and suppose that $(a,b) \ni r \mapsto t_r^+(u)$ (respectively $t_r^-(u)$) is well defined. Then $(a,b) \ni r \mapsto t_r^+(u)$ (respectively $t_r^-(u)$) is C^1 in (a,b).*

From Lemma 4.4 it is easy to see that, for each $r > 0$, $\mathcal{N}_r$ given in (4.2) can be written also as

$$\mathcal{N}_r = \{r^{1/2}u : u \in S_1, \varphi'_{r,u}(1) = 0\}$$

which, in some sense, justifies the name of Nehari set. Moreover, it holds (see (4.3), (4.4) and (4.5)) that

$$\begin{aligned} \mathcal{N}_r^+ &= \{u \in \mathcal{N}_r : \varphi''_{r,u}(1) > 0\}, \\ \mathcal{N}_r^0 &= \{u \in \mathcal{N}_r : \varphi''_{r,u}(1) = 0\}, \\ \mathcal{N}_r^- &= \{u \in \mathcal{N}_r : \varphi''_{r,u}(1) < 0\}, \end{aligned} \tag{4.10}$$

and $\mathcal{N}_r = \mathcal{N}_r^+ \cup \mathcal{N}_r^0 \cup \mathcal{N}_r^-$.

Remark 2: Note that, given $u \in S_1$, t^* is a critical point of the fiber map $\varphi_{r,u}$ if and only if $r^{1/2}u^{t^*} \in \mathcal{N}_r$. Actually t^* is a minimum (respectively maximum or inflection) point of $\varphi_{r,u}$ if and only if $r^{1/2}u^{t^*} \in \mathcal{N}_r^+$ (respectively $\mathcal{N}_r^-$ or $\mathcal{N}_r^0$).

In the following we study deeply the sets $\mathcal{N}_r^+$ and $\mathcal{N}_r^-$.

4.2. $\mathcal{N}_r^+$ AND $\mathcal{N}_r^-$ AS NATURAL CONSTRAINTS. Let us start by defining, for $r>0$, the functionals

$$h(u) = \frac{1}{2}\int u^2 - \frac{r}{2}, \quad \text{for } u \in H^1(\mathbb{R}^3), \tag{4.11}$$
$$g(u) = \varphi'_{r,u}(1), \qquad \text{for } u \in S_1.$$

LEMMA 4.7: *Whenever nonempty, $\mathcal{N}_r^+$ and $\mathcal{N}_r^-$ are C^1 manifolds in $H^1(\mathbb{R}^3)$ of co-dimension* 2.

Proof. Let us show the proof for $\mathcal{N}_r^+$ since for $\mathcal{N}_r^-$ it is completely analogous.

The proof will follow once we prove that $h'(u) \neq 0$, $g'(u) \neq 0$ and $h'(u)$, $g'(u)$ are linearly independent for each $u \in \mathcal{N}_r^+$. In fact, $h'(u) \neq 0$ is straightforward. Suppose on the contrary that there exists $u \in \mathcal{N}_r^+$ and $c \in \mathbb{R}$ such that

$$g'(u) = ch'(u).$$

It follows that

$$-2\Delta u - cu + q\phi_u u - \frac{3(p-2)}{2}\lambda |u|^{p-2}u = 0.$$

From Propostion 4.1 we conclude that

$$\begin{cases} \int |\nabla u|^2 - \frac{3c}{2}r + \frac{5}{4}q\int \phi_u u^2 - \frac{9(p-2)}{2p}\lambda \int |u|^p = 0, \\ 2\int |\nabla u|^2 - cr + q\int \phi_u u^2 - \frac{3(p-2)}{2}\lambda \int |u|^p = 0, \\ \int |\nabla u|^2 + \frac{q}{4}\int \phi_u u^2 - \frac{3(p-2)}{2p}\lambda \int |u|^p = 0. \end{cases}$$

Let us set for brevity

$$A = \int |\nabla u|^2, \quad B = q\int \phi_u u^2, \quad C = \lambda \int |u|^p$$

and solve the system with respect to these variables. A simple calculation shows that it has a unique solution when $p \neq 3$, in which case

$$A = \frac{rc(8-3p)}{8(p-3)}, \quad B = \frac{rc(3p-10)}{2(p-3)}, \quad C = \frac{rcp}{6(p-2)(p-3)}.$$

We substitute A, C in $\varphi''_{r,u}(1)$ to conclude that

$$\varphi''_{r,u}(1) = 0,$$

and hence a contradiction. If $p = 3$ we have two cases: when $c \neq 0$, then the system has no solution, which is a contradiction. When $c = 0$, the system has the following solution

$$A = \frac{C}{4}, \quad B = C, \quad C > 0.$$

We substitute A, C in $\varphi''_{r,u}(1)$ to conclude that

$$\varphi''_{r,u}(1) = 0,$$

again a contradiction. From all these contradictions we conclude that $h'(u)$ and $g'(u)$ are linearly independent for each $u \in \mathcal{N}_r^+$. Moreover, a careful look at the previous calculations shows that $g'(u) = 0$ is impossible, since in that case we would have $c = 0$, which gives a contradiction in all cases. Therefore $g'(u) \neq 0$ and $\mathcal{N}_r^+$ is a C^1 manifold with co-dimension 2 in $H^1(\mathbb{R}^3)$. ∎

Now we prove that $\mathcal{N}_r^+$ and $\mathcal{N}_r^-$ are natural constraints for the energy functional E.

Lemma 4.8: *Assume that there exist $u \in \mathcal{N}_r^+ \cup \mathcal{N}_r^-$ and $\mu, \nu \in \mathbb{R}$ such that*

$$E'(u) = \mu h'(u) + \nu g'(u)$$

where h and g are given by (4.11). Then $\nu = 0$.

Proof. Indeed, applying Proposition 4.1 to the equation

$$E'(u) - \mu h'(u) - \nu g'(u) = 0$$

we conclude that

$$\frac{3}{2}(E'(u)u - \mu h'(u)u - \nu g'(u)u) - P(u) = 0.$$

Simple calculations shows that

$$\frac{3}{2}(E'(u)u - \mu h'(u)u - \nu g'(u)u) - P(u) = g(u) - \nu\varphi''_{r,u}(1),$$

which implies that $\nu\varphi''_{r,u}(1) = 0$, and hence $\nu = 0$. ∎

Lemma 4.7 and Lemma 4.8 are summarized in the next

Theorem 4.9: *Whenever nonempty, $\mathcal{N}_r^+$ and $\mathcal{N}_r^-$ are C^1 manifolds in $H^1(\mathbb{R}^3)$ of co-dimension 2 and natural constraints for E.*

The next step is then to see for which values of q, λ, p, r the sets $\mathcal{N}_r, \mathcal{N}_r^+, \mathcal{N}_r^-$ are non-empty. As a consequence of this study, we will be able to recover some results known in the literature by our unified approach.

5. Structure of $\mathcal{N}_r, \mathcal{N}_r^+$ and $\mathcal{N}_r^-$

The structure of $\mathcal{N}_r, \mathcal{N}_r^+$ and $\mathcal{N}_r^-$ strongly depends on the values of p and indeed different approaches are needed. The particular value $p = 3$ is treated separately.

5.1. The case $p \neq 3$ and Proof of Theorem 2.3. It is convenient to consider the cases $p \in (2, 8/3] \cup [10/3, 6]$ and $p \in (8/3, 10/3) \setminus \{3\}$.

5.1.1. *The case* $p \in (2, 8/3] \cup [10/3, 6)$. In this case we can give a simple description of $\mathcal{N}_r$. We prefer to state separately the limit cases $p = 8/3$ and $p = 10/3$.

Theorem 5.1: *Let* $r > 0$. *Then:*

(i) *if* $p \in (2, 8/3)$, *then* $\mathcal{N}_r = \mathcal{N}_r^+ \neq \emptyset$;
(ii) *if* $p \in (10/3, 6)$, *then* $\mathcal{N}_r = \mathcal{N}_r^- \neq \emptyset$.

Proof. The proof of (i) is a direct consequence of Proposition 4.5 item (I) since for each $u \in S_1$ we have that $r^{1/2} u^{t_r^+(u)} \in \mathcal{N}_r^+$. Similarly, the proof of (ii) is a direct consequence of Proposition 4.5 item (V), since for each $u \in S_1$ we have that $r^{1/2} u^{t_r^-(u)} \in \mathcal{N}_r^-$. ∎

Theorem 5.2: *Let* $r > 0$. *If* $p = 8/3$, *then* $\mathcal{N}_r = \mathcal{N}_r^+ \neq \emptyset$.

Proof. In fact, from Proposition 4.5 item (II) it is sufficient to prove that there exists $u \in S_1$ such that

$$\frac{r^2}{4} \int \phi_u u^2 - \frac{r^{p/2}}{p} \int |u|^p < 0.$$

If $\{u_n\} \subset S_1$ is the sequence given by Lemma 3.2, then

$$\lim_{n\to\infty} \left(\frac{r^2}{4} \int q\phi_{u_n} u_n^2 - \frac{r^{p/2}}{p} \lambda \int |u_n|^p \right) \leq \lim_{n\to\infty} \left(\frac{C_3}{n^{2/3}} \frac{r^2}{4} q - C_1 \frac{r^{p/2}}{p} \lambda \right) = -C_1 \frac{r^{p/2}}{p} \lambda.$$

Therefore for n sufficiently large, we have that $r^{1/2} u_n^{t_r^+(u_n)} \in \mathcal{N}_r^+$. ∎

Theorem 5.3: *Let* $r > 0$. *If* $p = 10/3$, *then* $\mathcal{N}_r \neq \emptyset$ *if and only if*

$$\frac{5}{3K_{\mathrm{GN}}} \frac{1}{\lambda} < r^{2/3}$$

(as usual K_{GN} *is the Gagliardo–Nirenberg constant as in* (4.6)*). Moreover, in this case we have* $\mathcal{N}_r = \mathcal{N}_r^-$.

Proof. By Proposition 4.5 item (IV) it is sufficient to estimate, for $u \in S_1$, the quantity

$$\frac{r}{2}\int |\nabla u|^2 - \frac{r^{p/2}}{p}\int |u|^p.$$

By the Gagliardo–Nirenberg inequality (4.6) we have that

$$\int |u|^p \le K_{\mathrm{GN}} \int |\nabla u|^2, \quad \forall u \in S_1,$$

where

$$K_{\mathrm{GN}} = \sup_{u\in S_1} \frac{\int |u|^p}{\int |\nabla u|^2}$$

It follows that

$$\begin{aligned}\frac{r}{2}\int |\nabla u|^2 - \frac{r^{p/2}}{p}\lambda\int |u|^p &\ge \frac{r}{2}\int |\nabla u|^2 - K_{\mathrm{GN}}\frac{r^{p/2}}{p}\lambda\int |\nabla u|^2 \\ &= r\int |\nabla u|^2\Big(\frac{1}{2} - \frac{3K_{\mathrm{GN}}}{10}r^{2/3}\lambda\Big).\end{aligned}$$

By definition of K_{GN}, there exists $u \in S_1$ with

$$\frac{r}{2}\int |\nabla u|^2 - \frac{r^{p/2}}{p}\lambda\int |u|^p < 0$$

if, and only if,

$$\frac{5}{3K_{\mathrm{GN}}}\frac{1}{\lambda} < r^{2/3},$$

in which case $r^{1/2}u^{t_r^-(u)} \in \mathcal{N}_r^-$. ∎

5.1.2. *The case* $p \in (8/3, 10/3) \setminus \{3\}$. In this case the description of $\mathcal{N}_r$ is more involved. We use the ideas introduced by Il'yasov [10]: for $r > 0$ and $u \in S_1$, consider the system (recall the definitions in Subsection 4.1)

$$\varphi_{r,u}(s) = \varphi'_{r,u}(s) = 0.$$

Since $p \in (8/3, 10/3) \setminus \{3\}$ we can solve it with respect to the variables s and r to obtain a unique solution, denoted hereafter by $(s_0(u), r_0(u))$, given by

$$s_0(u) = \Big(\frac{p}{3p-8}\frac{1}{r^{(p-2)/2}}\frac{\int |\nabla u|^2}{\lambda\int |u|^p}\Big)^{\frac{2}{3p-10}}$$

and

$$r_0(u) = \Big(\frac{2(10-3p)}{3p-8}\Big)^{\frac{3p-10}{4(p-3)}}\Big(\frac{p}{3p-8}\Big)^{\frac{1}{2(p-3)}}R_p(u),$$

where R_p is defined in Section 3. The following proposition is just a consequence of the definitions and makes clear that $p = 3$ is a threshold.

Proposition 5.4: *Assume that* $p \in (8/3, 10/3) \setminus \{3\}$.*Then for each* $u \in S_1$, *there exists a unique pair* $(s_0(u), r_0(u))$ *such that*

$$\varphi_{r_0(u),u}(s_0(u)) = \varphi'_{r_0(u),u}(s_0(u)) = 0.$$

Moreover:

(1) *If* $p \in (8/3, 3)$ *and* $r < r_0(u)$, *then* $\varphi_{r,u}(s_0(u)) < 0$ *and* $\varphi'_{r,u}(s_0(u)) = 0$, *while if* $r > r_0(u)$, *then* $\varphi_{r,u}(s_0(u)) > 0$ *and* $\varphi'_{r,u}(s_0(u)) = 0$.

(2) *If* $p \in (3, 10/3)$ *and* $r < r_0(u)$, *then* $\varphi_{r,u}(s_0(u)) > 0$ *and* $\varphi'_{r,u}(s_0(u)) = 0$, *while if* $r > r_0(u)$, *then* $\varphi_{r,u}(s_0(u)) < 0$ *and* $\varphi'_{r,u}(s_0(u)) = 0$.

Similarly, for $r > 0$ and $u \in S_1$ we consider the system

$$\varphi'_{r,u}(s) = \varphi''_{r,u}(s) = 0.$$

Again, since $p \neq 3$ (and $p \neq 10/3$), we can solve it with respect to the variables s and r to obtain a unique solution, hereafter denoted by $(s(u), r(u))$, given by

$$s(u) = \Big(\frac{4p}{3(p-2)(3p-8)} \frac{1}{r^{(p-2)/2}} \frac{\int |\nabla u|^2}{\lambda \int |u|^p}\Big)^{\frac{2}{3p-10}}$$

and

$$r(u) = \Big(4\frac{10-3p}{3p-8}\Big)^{\frac{3p-10}{4(p-3)}} \Big(\frac{4p}{3(p-2)(3p-8)}\Big)^{\frac{1}{2(p-3)}} R_p(u).$$

However note that the expression for $r(u)$ makes sense also for $p = 10/3$ and in this case

$$r(u) = \Big(\frac{5}{3}\Big)^{3/2} R_{10/3}(u). \tag{5.1}$$

Similarly to Proposition 5.4 we have:

Proposition 5.5: *Assume that* $p \in (8/3, 10/3) \setminus \{3\}$. *Then for each* $u \in S_1$, *there exists a unique pair* $(s(u), r(u))$ *such that*

$$\varphi'_{r(u),u}(s(u)) = \varphi''_{r(u),u}(s(u)) = 0.$$

Moreover:

(1) *If* $p \in (8/3, 3)$ *and* $r < r(u)$, *then* $\varphi'_{r,u}(s(u)) < 0$ *and* $\varphi''_{r,u}(s(u)) = 0$, *while if* $r > r(u)$, *then* $\varphi'_{r,u}(s(u)) > 0$ *and* $\varphi''_{r,u}(s(u)) = 0$.

(2) *If* $p \in (3, 10/3)$ *and* $r < r(u)$, *then* $\varphi'_{r,u}(s(u)) > 0$ *and* $\varphi''_{r,u}(s(u)) = 0$, *while if* $r > r(u)$, *then* $\varphi'_{r,u}(s(u)) < 0$ *and* $\varphi''_{r,u}(s(u)) = 0$.

Furthermore

Proposition 5.6: *For each $u \in S_1$ we have that:*

(i) *if $p \in (8/3, 3)$, then $r_0(u) < r(u)$;*
(ii) *if $p \in (3, 10/3)$, then $r_0(u) > r(u)$.*

Moreover:

(iii) *if $p \in (8/3, 3)$, then the functions $S_1 \ni u \mapsto r_0(u), r(u)$ are unbounded from above;*
(iv) *if $p \in (3, 10/3)$, then the functions $S_1 \ni u \mapsto r_0(u), r(u)$ are bounded away from zero.*

Proof. The proofs of (i) and (ii) are straightforward and the proofs of (iii) and (iv) are a consequence of Proposition 3.4. ∎

To treat the case $p \in (3, 10/3)$ we need also the numbers

$$r_0^* := \inf_{u \in S_1} r_0(u) \quad \text{and} \quad r^* := \inf_{u \in S_1} r(u). \tag{5.2}$$

Then the description of $\mathcal{N}_r, \mathcal{N}_r^+, \mathcal{N}_r^-$ is given.

Theorem 5.7: *The following hold:*

(i) *Suppose that $p \in (8/3, 3)$. Then for each $r > 0$ there exists $u \in S_1$ such that $\inf_{t>0} \varphi_{r,u}(t) < 0$. Moreover, $\mathcal{N}_r^+$ and $\mathcal{N}_r^-$ are non-empty.*
(ii) *Suppose that $p \in (3, 10/3)$. If $r < r^*$, then $\mathcal{N}_r = \emptyset$, while if $r > r^*$, then $\mathcal{N}_r^+$ and $\mathcal{N}_r^-$ are non-empty. Moreover, if $r < r_0^*$, then $\inf_{t>0} \varphi_{r,u}(t) \geq 0$ for each $u \in S_1$, while if $r > r_0^*$, then there exists $u \in S_1$ such that $\inf_{t>0} \varphi_{r,u}(t) < 0$.*

Proof. (i) Fix $r > 0$ and assume on the contrary that for each $u \in S_1$ we have that $\inf_{t>0} \varphi_{r,u}(t) \geq 0$. In particular, it follows that $\varphi_{r,u}(t_r^+(u)) \geq 0$ and therefore, from Proposition 5.4 we conclude that $r_0(u) \leq r$ for all $u \in S_1$. This contradicts Proposition 5.6 (iii) and hence there exists $u \in S_1$ such that $\inf_{t>0} \varphi_{r,u}(t) < 0$. To conclude, note from Proposition 4.5 item (III) that if $u \in S_1$ satisfies $\inf_{t>0} \varphi_{r,u}(t) < 0$, then $r^{1/2} u^{t_r^-(u)} \in \mathcal{N}_r^-$ and $r^{1/2} u^{t_r^+(u)} \in \mathcal{N}_r^+$.

(ii) Fix $r < r^*$ and suppose on the contrary that $\mathcal{N}_r \neq \emptyset$. Take $u \in \mathcal{N}_r$ and observe from Proposition 4.5 item (III) that there exists $\overline{t} > 0$ such that $\varphi'_{r,u}(\overline{t}) \leq 0$ and $\varphi''_{r,u}(\overline{t}) = 0$. From Proposition 5.5 we conclude that $r \geq r(u) \geq r^*$ which is clearly a contradiction and therefore $\mathcal{N}_r = \emptyset$.

Now fix $r > r^*$ and assume on the contrary that $\mathcal{N}_r^+ = \emptyset$, which implies from Proposition 4.5 item (III) that $\mathcal{N}_r^- = \emptyset$ (and vice-versa). From the same proposition, we conclude that for each $u \in S_1$, when $\varphi''_{r,u}(t) = 0$ then $\varphi'_{r,u}(t) \geq 0$. It follows from Proposition 5.5 that $r < r(u)$ for all $u \in S_1$, again a contradiction and hence $\mathcal{N}_r^+$ and $\mathcal{N}_r^-$ are non-empty.

By using the functional $S_1 \ni u \mapsto r_0(u)$ instead of $S_1 \ni u \mapsto r(u)$, the rest of the proof is similar. ∎

Now we can give the proof of Theorem 2.3.

Indeed (i) follows by Theorem 5.1 and Proposition 4.5 item (I). (ii) follows by Theorem 5.1 and Proposition 4.5 item (V). (iii) follows by Theorem 5.2. (iv) follows by Theorem 5.3. (v) and (vi) follow by Theorem 5.7.

5.2. The case $p = 3$ and Proof of Theorem 2.4. In this case the system

$$\varphi_{r,u}(t) = \varphi'_{r,u}(t) = 0$$

has no solution with respect to the variables t, r. Therefore, instead of the variable r, we will solve the system with respect to the variable λ and analyze the dependence of the solutions with respect to q. It will be clear from the calculations that, at least topologically speaking, there are no changes in the fibering maps with respect to r, hence, to reflect the dependence on q, λ, we change the notation here, so for example we will write $\mathcal{N}_{q,\lambda}, \varphi_{q,\lambda,u}, \dots$ instead of $\mathcal{N}_r, \varphi_{r,u}, \dots$ we used up to now.

Consider then the system of equations $\varphi_{q,\lambda,u}(t) = \varphi'_{q,\lambda,u}(t) = 0$. We solve this system with respect to the variables t, λ to find a unique solution given by

$$t_{0,q}(u) = \Big(\frac{3}{r^{1/2}} \frac{\int |\nabla u|^2}{\lambda \int |u|^3}\Big)^{-2}$$

and

$$\lambda_{0,q}(u) = \Big(\frac{9}{2}\Big)^{\frac{1}{2}} q^{\frac{1}{2}} \frac{(\int |\nabla u|^2 \int \phi_u u^2)^{\frac{1}{2}}}{\int |u|^3}.$$

Similarly, we consider the system $\varphi'_{q,\lambda,u}(t) = \varphi''_{q,\lambda,u}(t) = 0$ and solve it with respect to the variables t and λ to obtain a unique solution given by

$$t_q(u) = \Big(\frac{4}{r^{1/2}} \frac{\int |\nabla u|^2}{\lambda \int |u|^3}\Big)^{-2}$$

and

$$\lambda_q(u) = 2q^{\frac{1}{2}} \frac{(\int |\nabla u|^2 \int \phi_u u^2)^{\frac{1}{2}}}{\int |u|^3}.$$

As an application of Lemma 3.3 we have:

PROPOSITION 5.8: *For each $r, q > 0$, the functionals $S_1 \ni u \mapsto \lambda_{0,q}(u), \lambda_q(u)$ are bounded away from zero. Moreover, $\lambda_q(u) < \lambda_{0,q}(u)$ for all $u \in S_1$.*

For each $r, q > 0$ define

$$\lambda_{0,q}^* := \inf_{u \in S_1} \lambda_{0,q}(u) \quad \text{and} \quad \lambda_q^* := \inf_{u \in S_1} \lambda_q(u).$$

Then the proof of Theorem 2.4 can be finished. Indeed it is similar to the proof of Theorem 5.7 (we use Proposition 5.8 instead of Proposition 5.6).

6. On the sub-additive property for $p \in (2, 10/3)$

For each $p \in (2, 10/3)$ define

$$I_r := I_{r,q,\lambda} = \inf\{E(u) : u \in \mathcal{N}_r^+ \cup \mathcal{N}_r^0\}.$$

Since E is bounded from below on S_r (see, e.g., [2, Lemma 3.1]) and $\mathcal{N}_r \subset S_r$, we conclude from Theorem 2.3 that I_r is well defined, that is $I_r > -\infty$.

In this Section we show how our method can be used to prove the sub-additive condition for I_r, namely

$$I_r < I_s + I_{r-s}, \quad 0 < s < r. \tag{6.1}$$

Again it is convenient to study separately the case $p = 3$.

6.1. THE CASE $p \in (2, 10/3) \setminus \{3\}$. We recall that, given $u \in S_1$, by definition $(\widetilde{r}(u), \widetilde{t}(u))$ is the unique solution of

$$\begin{cases} rt^2 \int |\nabla u|^2 + \frac{r^2 t}{4} q \int \phi_u u^2 - \frac{3(p-2)}{2p} r^{p/2} t^{\frac{3(p-2)}{2}} \lambda \int |u|^p = 0, \\ \frac{q}{2} r^2 t \int \phi_u u^2 - \frac{p-2}{p} r^{p/2} t^{\frac{3(p-2)}{2}} \lambda \int |u|^p = 0; \end{cases}$$

see (3.7) and (3.8) for the explicit value of the solutions.

PROPOSITION 6.1: *For each $p \in (2, 10/3) \setminus \{3\}$, the functional $S_1 \ni u \mapsto \widetilde{r}(u)$ is bounded away from 0. Moreover,*

(i) *if $p \in (3, 10/3)$, then $r(u) < \widetilde{r}(u) < r_0(u)$, for all $u \in S_1$;*
(ii) *if $p \in (8/3, 3)$, then $\widetilde{r}(u) < r_0(u) < r(u)$, for all $u \in S_1$.*

Proof. That $S_1 \ni u \mapsto \widetilde{r}(u)$ is bounded away from 0, for all $p \in (2, 10/3) \setminus \{3\}$, follows from Proposition 3.4 and Theorem 3.7. The proofs of (i) and (ii) are straightforward. ∎

As was already observed (see, e.g., [3]), in order to prove the strict subadditive condition (6.1), it is sufficient to show that I_r/r is decreasing in r. Our strategy to prove that I_r/r is decreasing in r will be the following: we will construct paths that cross the Nehari manifolds when r varies and show that the energy restricted to these paths, divided by r, is decreasing. Then we will show that the function I_r/r will inherit this property for some specific values of r.

Fix $u \in S_1$ and

(i) if $p \in (2, 8/3)$, define $f(r) := \varphi_{r,u}(t_r^+(u))$ for all $r \in (0, \infty)$;

(ii) if $p = 8/3$ and $\frac{r^2}{4} \int \phi_u u^2 - \frac{3}{8} r^{4/3} \lambda \int |u|^{8/3} < 0$, define $f(r) := \varphi_{r,u}(t_r^+(u))$ for all $r \in (0, r(u))$ where, in this case, $r(u)$ is by definition the unique $r > 0$ for which

$$\frac{r^2}{4} \int \phi_u u^2 - \frac{3}{8} r^{4/3} \lambda \int |u|^{8/3} = 0;$$

(iii) if $p \in (8/3, 3)$, define $f(r) := \varphi_{r,u}(t_r^+(u))$ for all $r \in (0, r(u))$;

(iv) if $p \in (3, 10/3)$, define $f(r) = \varphi_{r,u}(t_r^+(u))$ for all $r \in (r(u), \infty)$.

Define also

$$g(r) := \frac{f(r)}{r}.$$

Clearly f, and consequently g, depend on $u \in S_1$.

Proposition 6.2: *Let $u \in S_1$.*

(i) *If $p \in (2, 8/3)$, then the function $(0, \infty) \ni r \mapsto g(r)$ is decreasing for all $r \in (0, \widetilde{r}(u))$ and increasing for all $r \in (\widetilde{r}(u), r(u))$.*

(ii) *If $p = 8/3$, then the function $(0, \infty) \ni r \mapsto g(r)$ is decreasing for all $r \in (0, \widetilde{r}(u))$ and increasing for $r \in (\widetilde{r}(u), r(u))$.*

(iii) *If $p \in (8/3, 3)$, then the function $(0, r(u)) \ni r \mapsto g(r)$ is decreasing for all $r \in (0, \widetilde{r}(u))$.*

(iv) *If $p \in (3, 10/3)$, then the function $(r(u), \infty) \ni r \mapsto g(r)$ is decreasing for all $r \in (\widetilde{r}(u), \infty)$.*

Proof. Indeed, from the definition of $f(r)$, it follows from Lemma 4.6 that g is C^1 and

$$g'(r) = r\varphi'_{r,u}(t_r^+(u)) + \frac{q}{2} t_r^+(u) \int \phi_u u^2 - \frac{p-2}{p} r^{p/2-2} t_r^+(u)^{\frac{3p}{2}-3} \lambda \int |u|^p.$$

For simplicity denote $t_r = t_r^+(u)$. It follows that $g'(r) = 0$ if, and only if,

$$\begin{cases} rt_r \int |\nabla u|^2 + \frac{r^2}{4} q \int \phi_u u^2 - \frac{3(p-2)}{2p} r^{p/2} t_r^{\frac{3p}{2}-4} \lambda \int |u|^p = 0, \\ \frac{q}{2} t_r \int \phi_u u^2 - \frac{p-2}{p} r^{p/2-2} t_r^{\frac{3p}{2}-3} \lambda \int |u|^p = 0, \end{cases} \tag{6.2}$$

which is equivalent to system (3.6). Fix $r > 0$ and define

$$h(t) := \frac{q}{2} \int \phi_u u^2 - \frac{p-2}{p} r^{\frac{p}{2}-2} t^{\frac{3p}{2}-4} \lambda \int |u|^p.$$

We consider two cases:

Case 1: $p = 8/3$.

Observe that the first equation of (3.6) has a unique solution t. By plugging this solution in the left-hand side of the second equation, which is exactly $th(t)$, the proof of (ii) is complete.

Case 2: $p \in (2, 10/3) \setminus \{8/3, 3\}$.

Note that the second equation of (3.6) has a unique solution t. By plugging this solution in the left-hand side of the first equation, which is exactly $\varphi'_{r,u}(t)$, we conclude, by using Proposition 4.5, the following:

(1) if $p \in (2, 8/3)$ and $r \in (0, \widetilde{r}(u))$, then $\varphi'_{r,u}(t) > 0$, while for $r \in (\widetilde{r}(u), r(u))$ we have that $\varphi'_{r,u}(t) < 0$;
(2) if $p \in (8/3, 3)$ and $r \in (0, \widetilde{r}(u))$, then $\varphi'_{r,u}(t) < 0$;
(3) if $p \in (3, 10/3)$ and $r \in (\widetilde{r}(u), \infty)$, then $\varphi'_{r,u}(t) < 0$.

Now we can prove (i), (iii) and (iv).

(i) If $r \in (0, \widetilde{r}(u))$, then from item (1), we conclude that $t > t_r$ and hence $h(t_r) < h(t) = 0$, while if $r \in (\widetilde{r}(u), r(u))$, then $t < t_r$ and hence $h(t_r) > h(t) = 0$.

(iii) If $r \in (0, \widetilde{r}(u))$, then from item (2), we conclude that $t < t_r$ and hence $h(t_r) < h(t) = 0$, that is $g'(r) < 0$.

(iv) If $r \in (\widetilde{r}(u), \infty)$, then from item (3), we conclude that $t < t_r$ and hence $h(t_r) < h(t) = 0$, that is $g'(r) < 0$. ∎

Let us define now

$$\mathcal{M}_r = \Big\{ \frac{u}{\|u\|_2} : u \in \mathcal{N}_r^+ \text{ and } E(u) < 0 \Big\}.$$

Lemma 6.3: *The following hold:*

(i) *if* $p \in (2, 3)$ *and* $0 < r_1 < r_2 < r^*$, *then* $\mathcal{M}_{r_1} = \mathcal{M}_{r_2} = S_1$;
(ii) *if* $p \in (3, 10/3)$ *and* $r^* < r_1 < r_2$, *then* $\mathcal{M}_{r_1} \subset \mathcal{M}_{r_2}$.

Proof. (i) Fix $0 < r < r^*$. Then, $\varphi_{r,u}$ satisfies Item (III)-(1) of Proposition 4.5 for all $u \in S_1$ and hence $\mathcal{M}_{r_1} = \mathcal{M}_{r_2} = S_1$.

(ii) Indeed, if $u \in \mathcal{M}_{r_1}$, then the fiber map $\varphi_{r_1,u}$ satisfies Item $III - 1$) of Proposition 4.5. From Proposition 5.5 it follows that $r(u) < r_1 < r_2$ and hence $\varphi_{r_2,u}$ also satisfies Item (III)-(1) of Proposition 4.5, which implies that $\mathcal{M}_{r_1} \subset \mathcal{M}_{r_2}$. ∎

Lemma 6.4: *Suppose that $p \in (3, 10/3)$ and let $r \in [a, b]$ where $r_0^* < a < b$. Then there exists a negative constant c_r such that $g'(r) < c_r$ for all $u \in \mathcal{M}_r$ and $r \in [a, b]$.*

Proof. In order to prove the lemma, it is sufficient to prove that the left-hand side of the second equation of system (6.2) is bounded from above by c for all $u \in \mathcal{M}_r$ and $r \in [a, b]$.

First observe from Proposition 6.1 that $\widetilde{r}(u) < r_0(u) < r$ for all $u \in \mathcal{M}_r$ and all $r \in [a, b]$ and hence, from Theorem 6.2, we conclude that $g'(r) < 0$ for all $u \in \mathcal{M}_r$ and $r \in [a, b]$. Now note that

$$g(r) = \varphi_{r,u}(t_r^+(u))/r = \varphi_{r,su}(t_r^+(su))/r$$

for all $s > 0$ and therefore, by choosing $s = 1/\|\nabla u\|_2$, we can assume that $\|\nabla u\|_2 = 1$ for all $u \in \mathcal{M}_r$.

Suppose on the contrary that there exists a sequence $\{u_n\} \subset \mathcal{M}_r$ with $\|\nabla u_n\|_2 = 1$ and corresponding sequences $t_n > 0$, $r_n \in [a, b]$ such that

$$\begin{cases} r_n t_n \int |\nabla u_n|^2 + \frac{r_n^2}{4} q \int \phi_{u_n} u_n^2 - \frac{3(p-2)}{2p} r_n^{p/2} t_n^{\frac{3p}{2}-4} \lambda \int |u_n|^p = 0, \\ \frac{q}{2} t_n \int \phi_{u_n} u_n^2 - \frac{p-2}{p} r_n^{p/2-2} t_n^{\frac{3p}{2}-3} \lambda \int |u_n|^p = o_n(1). \end{cases} \tag{6.3}$$

From Proposition 4.3 and the Gagliardo–Nirenberg inequality it follows that there exist positive constants c, d such that $c \le t_n \le d$ and $c \le \int |u_n|^p \le d$ for all n. Therefore from the second equation of (6.3) we obtain that

$$t_n = \Big(\frac{p}{2(p-2)} r_n^{\frac{4-p}{2}} \frac{q}{\lambda} \frac{\int \phi_{u_n} u_n^2}{\int |u_n|^p}\Big)^{\frac{2}{3p-8}} + o_n(1).$$

By plugging t_n in the first equation of (6.3) we conclude that $r_n = \widetilde{r}(u_n) + o_n(1)$ and hence $r_n = c r_0(u_n) + o_n(1) < c r_n + o_n(1)$ where $c \in (0, 1)$, which is a contradiction. Then there exists a negative constant c_r such that $g'(r) < c_r$ for all $u \in \mathcal{M}_r$. ∎

Since we do not have a priori estimates like in Proposition 4.3 for the case $p \in (2,3)$, a similar version of Lemma 6.4 for that case is not so immediate, however, if we control the term $\int |u|^p$, then we can prove the following:

Lemma 6.5: *Suppose that* $p \in (2,3)$ *and let* $r \in [a,b]$ *where* $0 < a < b < \inf_{u \in S_1} \widetilde{r}(u)$. *Fix* $d > 0$. *Then there exists a negative constant* c *such that* $g'(r) < c$ *for all* $u \in \mathcal{M}_r$ *satisfying* $\int |u^{t_r^+(u)}|^p \geq c$ *and all* $r \in [a,b]$.

Proof. In order to prove the lemma, it is sufficient to prove that the left hand side of the second equation of system (6.2) is bounded from above by c for all $u \in \mathcal{M}_r$ satisfying $\int |u^{t_r^+(u)}|^p \geq d$ and all $r \in [a,b]$. From Theorem 6.2, we have that $g'(r) < 0$ for all $u \in \mathcal{M}_r$. Now note that

$$g(r) = \varphi_{r,u}(t_r^+(u))/r = \varphi_{r,su}(t_r^+(su))/r$$

for all $s > 0$ and therefore, by choosing $s = 1/\|\nabla u\|_2$, we can assume that $\|\nabla u\|_2 = 1$ for all $u \in \mathcal{M}_r$ satisfying $\int |u^{t_r^+(u)}|^p \geq d$. Suppose on the contrary that there exists a sequence $\{u_n\} \subset \mathcal{M}_r$ satisfying $\|\nabla u_n\|_2 = 1$ and $\int |u_n^{t_r^+(u_n)}|^p \geq d$ and corresponding sequences $t_n > 0$, $r_n \in [a,b]$ such that

$$\begin{cases} r_n t_n \int |\nabla u_n|^2 + \frac{r_n^2}{4} q \int \phi_{u_n} u_n^2 - \frac{3(p-2)}{2p} r_n^{p/2} t_n^{\frac{3p}{2}-4} \lambda \int |u_n|^p = 0, \\ \frac{q}{2} t_n \int \phi_{u_n} u_n^2 - \frac{p-2}{p} r_n^{p/2-2} t_n^{\frac{3p}{2}-3} \lambda \int |u_n|^p = o_n(1). \end{cases}$$

Arguing as in the proof of Lemma 6.4 we conclude that

$$r_n = \widetilde{r}(u_n) + o_n(1) \geq \inf_{u \in S_1} \widetilde{r}(u) + o_n(1) > b + \varepsilon + o_n(1)$$

for some ε, which is a contradiction. The proof is complete. ∎

At this point we have the desired result on I_r/r.

Theorem 6.6: *The following hold:*

(i) *if* $p \in (2,3)$, *then the function* $(0, \inf_{u \in S_1}, \widetilde{r}(u)) \ni r \mapsto I_r/r$ *is decreasing;*
(ii) *if* $p \in (3, 10/3)$, *then the function* $(r_0^*, \infty) \ni r \mapsto I_r/r$ *is decreasing.*

Proof. (i) Fix $0 < r_1 < r_2 < \inf_{u \in S_1} \widetilde{r}(u) < r^*$ and let $\{u_n\} \subset \mathcal{N}_{r_1}^+$ be a minimizing sequence to I_{r_1}. Since every such sequence is non-vanishing, we can assume that $\int |u_n|^p \geq d$ for some positive constant d and all $r \in [r_1, r_2]$. From Lemma 6.3, Lemma 6.5 and the mean value theorem, we conclude that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\frac{I_{r_2}}{r_2} &\leq \frac{\varphi_{r_2,u_n}(t^+_{r_2}(u_n))}{r_2}\\ &= \frac{\varphi_{r_1,u_n}(t^+_{r_1}(u_n))}{r_1} + g'(\theta)(r_2 - r_1)\\ &< \frac{\varphi_{r_1,u_n}(t^+_{r_1}(u_n))}{r_1} + c(r_2 - r_1)\end{aligned}$$

where $\theta \in (r_1, r_2)$. As a consequence

$$\frac{I_{r_2}}{r_2} \leq \frac{I_{r_1}}{r_1} + c(r_2 - r_1),$$

and the proof of (i) is complete.

(ii) Fix $r_0^* < r_1 < r_2$ and note from Lemma 6.3, Lemma 6.4 and the mean value theorem that

$$\begin{aligned}\frac{I_{r_2}}{r_2} &\leq \frac{\varphi_{r_2,u}(t^+_{r_2}(u))}{r_2}\\ &= \frac{\varphi_{r_1,u}(t^+_{r_1}(u))}{r_1} + g'(\theta)(r_2 - r_1)\\ &< \frac{\varphi_{r_1,u}(t^+_{r_1}(u_n))}{r_1} + c(r_2 - r_1), \qquad \forall u \in \mathcal{M}_{r_1},\end{aligned}$$

where $\theta \in (r_1, r_2)$. As a consequence

$$\frac{I_{r_2}}{r_2} \leq \frac{I_{r_1}}{r_1} + c(r_2 - r_1),$$

and the proof of is complete. ∎

As an immediate consequence of Theorem 6.6 we have the sub-additivity inequality for I_r.

THEOREM 6.7: *The following hold:*

(i) *if* $p \in (2, 3)$, *then for each* $r_1, r_2 \in (0, \inf_{u \in S_1} \widetilde{r}(u))$, *with* $r_1 < r_2$, *we have that* $I_{r_2} < I_{r_1} + I_{r_2 - r_1}$;

(ii) *if* $p \in (3, 10/3)$, *then for each* $r_1, r_2 \in (r_0^*, \infty)$, *with* $r_1 < r_2$, *we have that* $I_{r_2} < I_{r_1} + I_{r_2 - r_1}$.

Remark 3: When $p \in (2, 8/3]$ we see from Theorem 6.2 that after $\widetilde{r}(u)$ the function g is increasing. This suggest that the same property may hold for I_r/r when r is big and this suggests that I_r may not satisfy the strict sub-additive property.

6.2. The case $p = 3$. We assume that $\lambda > \lambda_q^*$, which implies from Theorem 2.4 that $\mathcal{N}_r^+ \neq \emptyset$ for all $r > 0$. We define in this case

$$\mathcal{M}_r = \Big\{ \frac{u}{\|u\|_2} : u \in \mathcal{N}_r^+ \Big\}.$$

Since, as observed in Subsection 5.2, the system $\varphi'_{r,u}(t) = \varphi''_{r,u}(t)$ does not depend on $r > 0$, it follows that

Lemma 6.8: *We have that*

$$\mathcal{M}_r = \mathcal{M}_1, \quad \forall r > 0.$$

From Lemma 6.8 we conclude that if $u \in \mathcal{M}_1 \subset S_1$, then $t_r^+(u)$ is defined for all $r > 0$ and thus we can define $f(r) = \varphi_{r,u}(t_r^+(u))$.

Lemma 6.9: *For each $r > 0$ and $u \in \mathcal{M}_1$, we have that $f(r) = f(1)r^3$.*

Proof. Note that

$$\frac{f(r)}{r^3} = \frac{1}{2}\Big(\frac{t_r^+(u)}{r}\Big)^2 \int |\nabla u|^2 + \frac{1}{4}\frac{t_r^+(u)}{r} \int \phi_u u^2 - \frac{1}{3}\Big(\frac{t_r^+(u)}{r}\Big)^{\frac{3}{2}} \int |u|^3.$$

Since $ru^{t_r^+(u)} \in \mathcal{N}_r^+$, we also have that

$$\Big(\frac{t_r^+(u)}{r}\Big)^2 \int |\nabla u|^2 + \frac{1}{4}\frac{t_r^+(u)}{r} \int \phi_u u^2 - \frac{1}{2}\Big(\frac{t_r^+(u)}{r}\Big)^{\frac{3}{2}} \int |u|^3 = 0.$$

Therefore

$$\Big(\frac{f(r)}{r^3}\Big)' = 0, \quad \forall r > 0,$$

which completes the proof. ∎

Proposition 6.10: *For each $r > 0$, we have that $I_r = I_1 r^3$.*

Proof. For each $u \in \mathcal{M}_1$, we have from Lemma 6.9 that

$$\frac{\varphi_{r,u}(t_r^+(u))}{r^3} = \varphi_{1,u}(t_1^+(u)).$$

Therefore

$$\frac{I_r}{r^3} = \inf_{u \in \mathcal{M}_1} \Big\{ \frac{\varphi_{r,u}(t_r^+(u))}{r^3} \Big\} = \inf_{u \in \mathcal{M}_1} \varphi_{1,u}(t_1^+(u)) = I_1$$

and the proof is completed. ∎

Then we have also for $p=3$ the sub-additivity condition.

THEOREM 6.11: *Suppose that $\lambda > \lambda^*_{0,q}$. Then for each $0<r_1<r_2$, we have that*

$$I_{r_2} < I_{r_1} + I_{r_2-r_1}.$$

Proof. From Theorem 2.4 we know that $\lambda^*_q < \lambda^*_{0,q}$ and $I_1<0$, therefore the conclusion is a consequence of Proposition 6.10. ∎

7. Constrained minimization for $p \in (2, 10/3)\setminus\{3\}$

Now we turn our attention to the existence of minimizers: it is convenient to consider two cases according to the values of p:

- $p \in (2,3)$,
- $p \in (3, 10/3)$,

although the first case is almost done.

7.1. THE CASE $p \in (2,3)$ AND PROOF OF THEOREM 2.5. The proof follows immediately from Theorem 6.7.

7.2. THE CASE $p \in (3, 10/3)$ AND PROOF OF THEOREM 2.6. By the definitions (see (5.2)):

$$\forall r > r^*_0 : I_r = \inf_{S_r} E < 0 \quad \text{and} \quad I_{r^*_0} = \inf_{S_{r^*_0}} E = 0.$$

In both cases the existence of minimizers is already known (see [3, 7, 12] and also our Theorem 6.7). However as we will see $0 = \inf_{S_r} E < I_r$ if $r \in (r^*, r^*_0)$. Let us start with the following

THEOREM 7.1: *If $(r^*, +\infty) \ni r \mapsto I_r$ is decreasing, then for each $r \in (r^*, r^*_0)$ there exists $u \in \mathcal{N}^+_r \cup \mathcal{N}^0_r$ such that $I_r = E(u)$.*

Proof. In fact, let $\{u_n\} \subset \mathcal{N}^+_r \cup \mathcal{N}^0_r$ be a minimizing sequence to I_r. It follows from Proposition 4.3 that there exist positive constants c, C such that

$$c \le \|u_n\| \le C, \quad \forall n \in \mathbb{N},$$

and we conclude that $u_n \nrightarrow 0$ in $L^p(\mathbb{R}^3)$. So $\{u_n\}$ does not vanish and then, up to translations, there exists a subsequence, still denoted by $\{u_n\}$, that converges weakly in $H^1(\mathbb{R}^3)$, strongly in $L^2_{loc}(\mathbb{R}^3)$ and almost everywhere in $\mathbb{R}^3$, to some non-zero function $u \in H^1(\mathbb{R}^3)$.

From [26, Lemma 2.2], we conclude that

$$I_r = \lim_{n\to\infty} E(u_n) = E(u) + \lim_{n\to\infty} E(u_n - u). \tag{7.1}$$

Let, as usual,

$$Q(u) = \int |\nabla u|^2 + \frac{q}{4}\int \phi_u u^2 - \frac{3(p-2)}{2p}\lambda \int |u|^p = 0,$$

and note that

$$0 = \lim_{n\to\infty} Q(u_n) = Q(u) + \lim_{n\to\infty} Q(u_n - u) \tag{7.2}$$

and

$$\|u\|_2^2 = r - \lim_{n\to\infty} \|u_n - u\|_2^2. \tag{7.3}$$

We claim that $Q(u) \le 0$. On the contrary we would have from (7.2) that $Q(u_n - u) < 0$ for sufficiently large n. From Proposition 4.5, there exists $t_n > 0$ such that $(u_n - u)^{t_n} \in \mathcal{N}^+_{\|u_n - u\|_2^2}$ for large n. Once $E(u_n - u) < I_r$ from (7.1) and $\|u_n - u\|_2^2 < r$ from (7.3) for sufficiently large n, we conclude that

$$I_{\|u_n - u\|_2^2} < E((u_n - u)^{t_n}) < E(u_n - u) < I_r,$$

which contradicts the hypothesis that $(r^*, \infty) \ni r \mapsto I_r$ is decreasing and therefore $Q(u) \le 0$.

From Proposition 4.5 there exists $t > 0$ such that $u^t \in \mathcal{N}^+_{\|u\|_2^2} \cup \mathcal{N}^0_{\|u\|_2^2}$. Thus, since $E(u) \le I_r$ from (7.1) and $\|u\|_2^2 \le r$ from (7.3), we conclude that

$$I_{\|u\|_2^2} \le E(u^t) \le E(u) \le I_r.$$

Therefore, from the hypothesis $(r^*, \infty) \ni r \mapsto I_r$ is decreasing, we conclude that $r = \|u\|_2^2$, $u \in \mathcal{N}^+_{\|u\|_2^2} \cup \mathcal{N}^0_{\|u\|_2^2}$ and $E(u) = I_r$. ∎

In order to make use of Theorem 7.1, we need to show that $(r^*, +\infty) \ni r \mapsto I_r$ is decreasing. Unfortunately we are able to do so only for some values of $p \in (3, 10/3)$, although we conjecture it is true for all p in the range. We note here that in fact, when $I_r < 0$ this is a standard result in the literature. However, when $I_r > 0$, which is the case for $r \in [r^*, r_0^*]$ (see Theorem 2.3), the inequalities go in the opposite direction and thus the proof seems not to be direct.

Our strategy to prove that I_r is decreasing in r will be the following: we will construct paths that cross the Nehari manifolds when r varies and show that the energy restricted to these paths is decreasing. To this end we need to calculate some derivatives.

LEMMA 7.2: *If $r^{1/2}u^t \in \mathcal{N}_r^+$, then*

$$t\int|\nabla u|^2 + \frac{r}{2}q\int\phi_u u^2 - \frac{3(p-2)}{4}t^{\frac{3p}{2}-4}r^{\frac{p}{2}-1}\lambda\int|u|^p < 0.$$

Proof. For simplicity denote

$$A = \int|\nabla u|^2, \quad B = \int\phi_u u^2 \text{ and } C = \lambda\int|u|^p.$$

By assumption we have that

$$\begin{cases} rtA + r^2\frac{q}{4}B - \frac{3(p-2)}{2p}r^{\frac{p}{2}}t^{\frac{3p}{2}-4}\lambda C = 0, \\ rA - \frac{3(p-2)(3p-8)}{4p}r^{\frac{p}{2}}t^{\frac{3p}{2}-5}\lambda C > 0. \end{cases} \tag{7.4}$$

From the equality in (7.4) we conclude that

$$tA + \frac{r}{2}qB - \frac{3(p-2)}{4}t^{\frac{3p}{2}-4}r^{\frac{p}{2}-1}\lambda C = -tA + \frac{3(p-2)(4-p)}{4p}t^{\frac{3p}{2}-4}r^{\frac{p}{2}-1}\lambda C.$$

From the inequality of (7.4) we obtain

$$tA + \frac{r}{2}qB - \frac{3(p-2)}{4}t^{\frac{3p}{2}-4}r^{\frac{p}{2}-1}\lambda C < \frac{3(p-2)(3-p)}{p}t^{\frac{3p}{2}-4}r^{\frac{p}{2}-1}\lambda C < 0$$

which is the conclusion. ∎

COROLLARY 3: *Let $I \subset \mathbb{R}$ be an open interval and fix $u \in S_1$. If $t_r^+(u)$ is defined for all $r \in I$, then the function $I \ni r \mapsto t_r^+(u)$ is C^1.*

Proof. Indeed, define

$$F(r,t) = \varphi'_{r,u}(t)$$

for $r \in I$ and $t > 0$. From Lemma 7.2 it follows that $F(r, t_r^+(u)) = 0$ and $\frac{\partial F}{\partial r}(r, t_r^+(u)) < 0$. The proof is then a consequence of the Implicit Function Theorem. ∎

Consider the equation $-27x^2 + 146x - 192 = 0$. It has two real roots and the biggest one is given by

$$p_0 = \frac{73+\sqrt{145}}{27} \in (3, 10/3).$$

Lemma 7.3: *Assume that $r^{1/2}u^t \in \mathcal{N}_r^+$. The following hold:*

(i) *if $p \in (p_0, 10/3)$, then there exists a constant $c_p'' < 0$ such that*

$$t^2 \int |\nabla u|^2 + rtq \int \phi_u u^2 - \lambda r^{\frac{p}{2}-1} t^{\frac{3p}{2}-3} \int |u|^p < \frac{c_p''}{r^2};$$

(ii) *if $p = p_0$, then*

$$t^2 \int |\nabla u|^2 + rtq \int \phi_u u^2 - \lambda r^{\frac{p_0}{2}-1} t^{\frac{3p_0}{2}-3} \int |u|^{p_0} < 0.$$

Proof. For simplicity denote

$$A = \int |\nabla u|^2, \quad B = \int \phi_u u^2 \quad \text{and} \quad C = \lambda \int |u|^p.$$

By assumption (see Lemma 4.4 and (4.10)) we have that

$$\begin{cases} rtA + r^2 \frac{q}{4} B - \frac{3(p-2)}{2p} r^{\frac{p}{2}} t^{\frac{3p}{2}-4} \lambda C = 0, \\ rA - \frac{3(p-2)(3p-8)}{4p} r^{\frac{p}{2}} t^{\frac{3p}{2}-5} \lambda C > 0. \end{cases} \tag{7.5}$$

From the equality in (7.5) we conclude that

$$t^2 A + rtqB - \lambda r^{\frac{p}{2}-1} t^{\frac{3p}{2}-3} C = -3t^2 A + \frac{5p-12}{p} t^{\frac{3p}{2}-3} r^{\frac{p}{2}-1} \lambda C$$

and then, from the inequality in (7.5), we obtain

$$\begin{aligned} t^2 A + rtqB - \lambda r^{\frac{p}{2}-1} t^{\frac{3p}{2}-3} C & \\ &< \Big(\frac{-9(p-2)(3p-8)}{4p} + \frac{5p-12}{p} \Big) \lambda r^{\frac{p}{2}-1} t^{\frac{3p}{2}-3} C \\ &= \frac{-27p^2 + 146p - 192}{4p} \frac{\lambda}{r} \int |r^{1/2} u^t|^p. \end{aligned} \tag{7.6}$$

Since by assumption $r^{1/2}u^t \in \mathcal{N}_r^+$ (see Remark 2), from Proposition 4.3 there exists a constant $c_p' > 0$ such that

$$\int |r^{1/2} u^t|^p \geq \frac{c_p'}{\lambda r},$$

therefore, from the definition of p_0, coming back to (7.6), it follows that

$$t^2 \int |\nabla u|^2 + rtq \int \phi_u u^2 - \lambda r^{\frac{p}{2}-1} t^{\frac{3p}{2}-3} \int |u|^p < \frac{-27p^2 + 146p - 192}{4p} \frac{c_p'}{r^2} := \frac{c_p''}{r^2},$$

from which the conclusion easily follows. ∎

Observe that by (4.9), c_p'' has an explicit expression.

Proposition 7.4: *Suppose that $p \in (p_0, 10/3)$. Then the function*

$$(r^*, \infty) \ni r \mapsto I_r$$

is decreasing.

Proof. Define $f(r) = E(r^{1/2}u^{t_r^+(u)})$ and set for brevity $t(r) = t_r^+(u)$. Observe from Proposition 4.6 that f is differentiable and

$$f'(r) = \frac{t(r)^2}{2}\int |\nabla u|^2 + \frac{rt(r)}{2}q\int \phi_u u^2 - \frac{\lambda}{2}r^{\frac{p}{2}-1}t(r)^{\frac{3p}{2}-3}\int |u|^p.$$

From Lemma 7.3 we conclude that $f'(r) \leq 2c_p''/r^2$. Fix $r^* < r_1 < r_2$ and $u \in \mathcal{M}_{r_1}$. If $r \in [r_1, r_2]$ then $f'(r) \leq 2c_p''/r_1^2$. Therefore there exists $\theta \in (r_1, r_2)$ such that

$$f(r_2) - f(r_1) = f'(\theta)(r_2 - r_1) \leq \frac{2c_p''}{r_1^2}(r_2 - r_1),$$

and hence

$$E(r_2^{1/2}u^{t(r_2)}) \leq E(r_1^{1/2}u^{t(r_1)}) + \frac{2c_p''}{r_1^2}(r_2 - r_1),$$

which implies that $I_{r_2} < I_{r_1}$ and the proof is finished. ∎

As a consequence of Theorem 7.1 and Proposition 7.4 we have:

Theorem 7.5: *Fix $p \in (p_0, 10/3)$. Then for each $r \in (r^*, r_0^*)$ there exists $u \in \mathcal{N}_r^+ \cup \mathcal{N}_r^0$ such that $I_r = E(u)$.*

Now we will show that for r near r_0^* the minimizer found in Theorem 7.5 belongs to $\mathcal{N}_r^+$. To this end we need to compare the energy of E restricted to $\mathcal{N}_r^0$ with I_r.

Lemma 7.6: *For each $r > r^*$, there exists a positive constant c such that*

$$E(u) \geq \frac{c}{r}, \quad \forall u \in \mathcal{N}_r^0.$$

Proof. Indeed by using the pair of equations that characterize $u \in \mathcal{N}_r^0$, that is

$$\begin{cases} rtA + r^2\frac{q}{4}B - \frac{3(p-2)}{2p}r^{\frac{p}{2}}t^{\frac{3p}{2}-4}\lambda C = 0, \\ rA - \frac{3(p-2)(3p-8)}{4p}r^{\frac{p}{2}}t^{\frac{3p}{2}-5}\lambda C = 0, \end{cases}$$

we can deduce that

$$E(u) = \frac{10-3p}{6(p-2)}\int |\nabla u|^2, \quad u \in \mathcal{N}_r^0,$$

therefore from Proposition 4.3 the proof is complete. ∎

LEMMA 7.7: *We have that*

$$\lim_{r\uparrow r_0^*} I_r = 0.$$

Proof. As we observed before $E_{r_0^*}=I_{r_0^*}$ and there exists $w\in\mathcal{N}_{r_0^*}^+$ with $E(w)=I_{r_0^*}$. Let $u \in S_1$ be such that $w = r_0^* u^{t_{r_0^*}^+(u)}$. Since $E(w) = 0$ we conclude from the definition of r_0^* and Theorem 5.7 that $r_0(u) = r_0^*$. Moreover, $r^* = r(u)$. It follows that $t_r^+(u)$ is well-defined for each $r \in (r^*, r_0^*)$. Since $I_r \geq 0$ for all $r \in (r^*, r_0^*)$ we obtain from Corollary 3 that

$$0 = \lim_{r\uparrow r_0^*} E(ru^{t_r^+(u)}) \geq \lim_{r\uparrow r_0^*} I_r \geq 0$$

and the proof is concluded. ∎

THEOREM 7.8: *For each $p \in (p_0, 10/3)$ there exists $\varepsilon > 0$ such that for each $r \in (r_0^* - \varepsilon, r_0^*)$, I_r is achieved. More specifically, there exists $u \in \mathcal{N}_r^+$ satisfying*

$$I_r = E(u).$$

Proof. From Theorem 7.5 it remains to prove that $u \in \mathcal{N}_r^+$. Let c/r be the constant given by Lemma 7.6. Given $0 < d < c/r$, from Lemma 7.7 there exists $\varepsilon > 0$ such that $I_r < d$ for all $r \in (r_0^* - \varepsilon, r_0^*)$. In particular, since $I_r < c/r$ it follows that $u \notin \mathcal{N}_r^0$ for all $r \in (r_0^* - \varepsilon, r_0^*)$ and consequently $u \in \mathcal{N}_r^+$. ∎

We can finish now the proof of Theorem 2.6. In fact (i) follows by Proposition 7.4 and Lemma 7.7; (ii) follows by Theorem 7.5 and (iii) follows by Theorem 7.8.

8. The case $p \in [10/3, 6)$

This case was treated in [1], where existence of global minimizers over the Nehari manifold $\mathcal{N}_r^-$ was proved for small r. Their proof relies on the fact that for small r the function

$$J_r = \inf\{E(u) : u \in \mathcal{N}_r^-\}$$

is decreasing.

Fix $u \in S_1$ and consider the map

$$f(r) := \varphi_{r,u}(t_r^-(u)), \tag{8.1}$$

where $t_r^-(u)$ is the unique critical point of $\varphi_{r,u}$. By Proposition 4.5, the map f is well-defined,

(i) for all $r \in (r(u), \infty)$, if $p = 10/3$ and $\frac{r}{2}\int |\nabla u|^2 - \frac{r^{p/2}}{p}\lambda \int |u|^p < 0$, in this case $r(u)$ being the unique $r > 0$ for which $\frac{r}{2}\int |\nabla u|^2 - \frac{r^{p/2}}{p}\lambda \int |u|^p = 0$, it coincides exactly with the value given in (5.1), justifying then the same notation;

(ii) for all $r \in (0, \infty)$, if $p \in (10/3, 6)$.

Now observe from Lemma 4.6 that (in both cases) f is C^1 and

$$\begin{aligned} f'(r) =& \varphi'_{r,u}(t_r^-(u)) \\ &+ \frac{1}{2}\left(t_r^-(u)^2 \int |\nabla u|^2 + r t_r^-(u) q \int \phi_u u^2 - \lambda r^{\frac{p}{2}-1} t_r^-(u)^{\frac{3p}{2}-3} \int |u|^p\right) \\ =& \frac{1}{2}\left(t_r^-(u)^2 \int |\nabla u|^2 + r t_r^-(u) q \int \phi_u u^2 - \lambda r^{\frac{p}{2}-1} t_r^-(u)^{\frac{3p}{2}-3} \int |u|^p\right), \end{aligned}$$

being (see Lemma 4.4)

$$\begin{aligned} 0 &= \varphi'_{r,u}(t_r^-(u)) \\ &= t_r^-(u) \int |\nabla u|^2 + \frac{r^2}{4} q \int \phi_u u^2 - \frac{3(p-2)}{2p} r^{p/2} t_r^-(u)^{\frac{3p}{2}-4} \lambda \int |u|^p. \end{aligned}$$

For simplicity denote $t_r = t_r^-(u)$. It follows that $f'(r) = 0$ if, and only if,

$$\begin{cases} r t_r \int |\nabla u|^2 + \frac{r^2}{4} q \int \phi_u u^2 - \frac{3(p-2)}{2p} r^{p/2} t_r^{\frac{3p}{2}-4} \lambda \int |u|^p = 0, \\ t_r^2 \int |\nabla u|^2 + r t_r q \int \phi_u u^2 - \lambda r^{\frac{p}{2}-1} t_r^{\frac{3p}{2}-3} \int |u|^p = 0. \end{cases} \tag{8.2}$$

From Proposition 3.1, system (8.2) has a unique solution $(\overline{r}(u), \overline{t}(u))$, where

$$\overline{r}(u) = \left(\frac{2(6-p)}{5p-12}\right)^{\frac{3p-10}{4(p-3)}} \left(\frac{3p}{5p-12}\right)^{\frac{1}{2(p-3)}} R_p(u).$$

Note that $\overline{r}(u) = (\frac{9}{7})^{3/2} r(u)$ when $p = 10/3$ and that, by Proposition 3.6, we have that

$$\inf_{u \in S_1} \overline{r}(u) > 0.$$

Theorem 8.1: *Suppose that* $u \in S_1$.

(i) *If* $p = 10/3$, *then the function* $(\overline{r}(u), \infty) \ni r \mapsto f(r)$ *is increasing.*
(ii) *If* $p \in (10/3, 6)$, *then the function* $(0, \infty) \ni r \mapsto f(r)$ *is decreasing for all* $r \in (0, \overline{r}(u))$ *and increasing for* $r \in (\overline{r}(u), \infty)$.

Proof. Let $t_r = t_r^-(u)$. By multiplying the first equation of (8.2) by -4 and substituting into the second one, we obtain that

$$t_r^2 \int |\nabla u|^2 + r t_r q \int \phi_u u^2 - \lambda r^{\frac{p}{2}-1} t_r^{\frac{3p}{2}-3} \int |u|^p = t_r^2 h(r), \tag{8.3}$$

where

$$h(r) = -3 \int |\nabla u|^2 + \frac{5p-12}{p} \lambda r^{\frac{p}{2}-1} t_r^{\frac{3p-10}{2}} \int |u|^p.$$

Since (8.3) means that $2f'(r) = t_r^2 h(r)$, we are reduced to studying the sign of h.

(i) This item is direct since, for $p = 10/3$, $h(r) > 0$ for

$$r > \left(\frac{15}{7}\right)^{3/2} R_{10/3} = \left(\frac{9}{7}\right)^{3/2} r(u) = \overline{r}(u).$$

(ii) We show that $f'(r)$ is negative for $r < \overline{r}(u)$ and positive for $r > \overline{r}(u)$. Taking into account that t_r is continuous (see Lemma 4.6) and f has a unique critical point since the solution of (8.2) is unique, it is sufficient to show that there exist some $0 < r_1 < \overline{r}(u) < r_2$ such that $h(r_1) < 0$ and $h(r_2) > 0$.

We start with the existence of r_1. We claim that

$$\lim_{r \to 0} h(r) < 0. \tag{8.4}$$

Indeed, if t_r is bounded from above as $r \to 0$, then (8.4) is obvious; therefore let us assume that $t_r \to \infty$ as $r \to 0$. Note from the first equation of (8.2) that

$$\int |\nabla u|^2 - \frac{3(p-2)}{2p} r^{p/2-1} t_r^{\frac{3p-10}{2}} \lambda \int |u|^p = o_r(1),$$

and hence

$$r^{p/2-1} t_r^{\frac{3p-10}{2}} \lambda \int |u|^p = \frac{2p}{3(p-2)} \int |\nabla u|^2 + o_r(1).$$

Then

$$\begin{aligned} h(r) &= -3\int |\nabla u|^2 + \frac{5p-12}{p}\lambda r^{\frac{p}{2}-1} t_r^{\frac{3p-10}{2}} \int |u|^p, \\ &= -3\int |\nabla u|^2 + \frac{5p-12}{p}\frac{2p}{3(p-2)}\int |\nabla u|^2 + o_r(1), \\ &= \frac{p-6}{3p-6}\int |\nabla u|^2 + o_r(1), \quad \text{as } r\to 0. \end{aligned}$$

Therefore the claim is proved.

Now we prove the existence of r_2. We claim that

$$\lim_{r\to+\infty} h(r) > 0. \tag{8.5}$$

Indeed, if t_r is bounded away from 0 as $r \to 0$, then (8.5) is obvious; therefore let us assume that $t_r \to 0$ as $r \to \infty$. Note from the first equation of (8.2) that

$$\int |\nabla u|^2 - \frac{3(p-2)}{2p} r^{p/2-1} t_r^{\frac{3p-10}{2}}\lambda \int |u|^p = -\frac{r}{4t_r} q \int \phi_u u^2.$$

Since $r/4t_r \to +\infty$ as $r \to +\infty$, we conclude that $r^{p/2-1} t_r^{\frac{3p-10}{2}} \lambda \int |u|^p \to +\infty$ as $r \to +\infty$ and the proof is complete. ∎

Let $p \in (10/3, 6)$ and, for $r, c, d > 0$, define

$$\mathcal{M}_r = \left\{ \frac{u}{\|u\|_2} : u \in \mathcal{N}_r^- \text{ and } \int |u|^p \geq c \text{ and } \int |\nabla u|^2 \leq d \right\}.$$

Lemma 8.2: *Suppose that $p \in (10/3, 6)$, $r \in [a, b]$ where $0 < a < b < \inf_{u\in S_1} \overline{r}(u)$ and $c, d > 0$. Then there exists a negative constant $\widetilde{c} = \widetilde{c}(a, b, r, c, d)$ such that $f'(r) < \widetilde{c}$ for all $u \in \mathcal{M}_r$ and all $r \in [a, b]$.*

Proof. In order to prove the lemma, it is sufficient to prove that the left hand side of the second equation of system (8.2) is bounded from above by $\widetilde{c}$ for all $u \in \mathcal{M}_r$ and $r \in [a, b]$. From Theorem 8.1, we have that $f'(r) < 0$ for all $u \in S_1$. Now note that $f(r) = \varphi_{r,u}(t_r^-(u)) = \varphi_{r,su}(t_r^-(su))$ for all $s > 0$ and therefore, by choosing $s = 1/\|\nabla u\|_2$, we can assume that $\|\nabla u\|_2 = 1$ for all $u \in \mathcal{M}_r$. Suppose on the contrary that there exists a sequence $\{u_n\} \subset \mathcal{M}_r$ satisfying $\|\nabla u_n\|_2 = 1$ and corresponding sequences $\{t_n\} \subset (0, +\infty)$, $\{r_n\} \subset [a, b]$ such that

$$\begin{cases} rt_n \int |\nabla u_n|^2 + \frac{r^2}{4} q \int \phi_{u_n} u_n^2 - \frac{3(p-2)}{2p} r^{p/2} t_n^{\frac{3p}{2}-4} \lambda \int |u_n|^p = 0, \\ t_n^2 \int |\nabla u_n|^2 + rt_n q \int \phi_{u_n} u_n^2 - \lambda r^{\frac{p}{2}-1} t_n^{\frac{3p}{2}-3} \int |u_n|^p = o_n(1). \end{cases}$$

From Proposition 4.3 and the condition $\int |\nabla u_n^{t_n}|^2 \leq d$ we conclude that $\{t_n\}$ is a bounded sequence which is also bounded away from zero; therefore $\int |u_n|^p$ is bounded away from zero and, arguing as in the proof of Lemma 6.4, we conclude that $r_n = \overline{r}(u_n) + o_n(1) \geq \inf_{u\in S_1} \overline{r}(u) + o_n(1) > b + \varepsilon + o_n(1)$ for some ε, which is a contradiction. The proof is complete. ∎

Lemma 8.3: *Suppose that $p \in (10/3, 6)$. Then $J_r > 0$ and every minimizing sequence is bounded and non-vanishing.*

Proof. Note that

$$E(u) = \frac{3p-10}{6(p-2)} \int |\nabla u|^2 + \frac{3p-8}{12(p-2)} \int \phi_u u^2, \quad \forall u \in \mathcal{N}_r^-. \tag{8.6}$$

Therefore from Proposition 4.3 we deduce that $J_r > 0$. If $\{u_n\}$ is a minimizing sequence, then from (8.6) we conclude that $\{\|\nabla u_n\|_2\}$ is bounded and hence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Moreover, this sequence can not vanish, since on the contrary, we would obtain from the equation

$$\int |\nabla u_n|^2 + \frac{q}{4} \int \phi_{u_n} u_n^2 - \frac{3(p-2)}{2p} \lambda \int |u_n|^p = 0$$

that $\int |\nabla u_n|^2 \to 0$ which contradicts $J_r > 0$. ∎

Theorem 8.4: *The function $(0,\infty) \ni r \mapsto J_r$ is decreasing over the interval $(0, \inf_{u\in S_1} \overline{r}(u))$.*

Proof. Fix $0 < r_1 < r_2 < \inf_{u\in S_1}, \overline{r}(u) < r^*$ and let $\{u_n\} \subset \mathcal{N}_{r_1}^+$ be a minimizing sequence to I_{r_1}. From the mean value theorem we have that

$$J_{r_2} \leq \varphi_{r_2,u_n}(t_{r_2}^-(u_n)) = \varphi_{r_1,u_n}(t_{r_1}^-(u_n)) + f'(\theta_n)(r_2 - r_1), \quad \forall n \in \mathbb{N},$$

where $\theta_n \in (r_1, r_2)$. From Lemma 8.3, $\{u_n\}$ being bounded and non-vanishing, it follows that $\{u_n/\|u_n\|_2\} \subset \mathcal{M}_{r_1}$ and therefore $\{u_n/\|u_n\|_2\} \subset \mathcal{M}_r$ for all $r \in [r_1, r_2]$. From Lemma 8.2 we conclude that $f'(\theta_n)(r_2 - r_1) < c(r_2 - r_1)$ where $c < 0$. As a consequence

$$J_{r_2} \leq J_{r_1} + c(r_2 - r_1),$$

and the proof is complete. ∎

Now we can conclude the proof of our last theorem.

8.1. Proof of Theorem 2.7. Since by the previous theorem J_r is decreasing, the result follows by [1].

Appendix A. New inequalities

We conclude with some estimates; in particular, the second one is new in the literature.

Theorem A.1: *The following hold:*

(i) *for each* $p \in (2,3)$, *there exists a positive function* $\mathfrak{f}_p : (0,\infty)^2 \to \mathbb{R}$ *such that if* $r_1 < r_2$, *then*

$$I_{r_2} > \Big(\frac{r_2}{r_1}\Big)^3 I_{r_1} + \mathfrak{f}_p(r_1,r_2)\lambda\Big(\frac{r_1}{r_2}\Big)^{p-3}\Big[\Big(\frac{r_1}{r_2}\Big)^{2(p-3)} - 1\Big];$$

(ii) *for each* $p \in (3,10/3)$ *and* $r^* < r_1 < r_2$, *then*

$$I_{r_2} < \Big(\frac{r_2}{r_1}\Big)^3 I_{r_1} - \frac{c_p'}{r_1}\Big(\frac{r_2}{r_1}\Big)^p\Big[\Big(\frac{r_2}{r_1}\Big)^{2(p-3)} - 1\Big],$$

where $c_p' > 0$ *is the constant given in Proposition 4.3.*

Proof. (i) Indeed, fix $r_1 < r_2$ and take $u \in \mathcal{M}_{r_2}$ satisfying $E(r_2^{1/2}u^{t(r_2)}) < 0$. For simplicity we set $t_i = t(r_i)$, $i = 1,2$. From Lemma 6.3 we know that $u \in \mathcal{M}_{r_1}$ and $E(r_1^{1/2}u^{t_1}) < 0$, which implies that t_1 is a global minimum for the fiber map $\varphi_{r_1,u}$ and therefore

$$\begin{aligned} E(r_2^{1/2}u^{t_2}) &= \frac{r_2^3}{r_1^3}E\Big(r_1^{1/2}u^{t_2\frac{r_1}{r_2}}\Big) + \frac{\lambda}{p}\Big(\frac{r_1}{r_2}\Big)^{p-3}\Big[\Big(\frac{r_1}{r_2}\Big)^{2(p-3)} - 1\Big]\int |r_2^{1/2}u^{t_2}|^p, \\ &> \frac{r_2^3}{r_1^3}E(r_1^{1/2}u^{t_1}) + \frac{\lambda}{p}\Big(\frac{r_1}{r_2}\Big)^{p-3}\Big[\Big(\frac{r_1}{r_2}\Big)^{2(p-3)} - 1\Big]\int |r_2^{1/2}u^{t_2}|^p. \end{aligned}$$

Since $p \in (2,3)$, it follows that $(r_1/r_2)^{2(p-3)} - 1 > 0$, therefore if $\{u_n\} \subset \mathcal{M}_{r_2}$ is choosen in such a way that $\{r_2^{1/2}u_n^{t_2}\}$ is a minimizing sequence for I_{r_2}, since it must be non-vanishing we obtain that

$$I_{r_2} > \frac{r_2^3}{r_1^3}I_{r_1} + \mathfrak{f}_p(r_1,r_2)\lambda\Big(\frac{r_1}{r_2}\Big)^{p-3}\Big[\Big(\frac{r_1}{r_2}\Big)^{2(p-3)} - 1\Big].$$

(ii) Indeed, fix $r^* < r_1 < r_2$ and take $u \in \mathcal{M}_{r_1}$. For simplicity, let again $t_i = t(r_i)$ for $i = 1, 2$ and set

$$Q(u) := \int |\nabla u|^2 + \frac{q}{4}\int \phi_u u^2 - \frac{3(p-2)}{2p}\lambda \int |u|^p.$$

Observe that

$$\begin{aligned} Q(r_1^{1/2}u^{t_1}) &= t_1 r_1 \int |\nabla u|^2 + \frac{r_1^2}{4} q \int \phi_u u^2 - \frac{3(p-2)}{2p} t_1^{\frac{3p}{2}-4} r_1^{\frac{p}{2}} \lambda \int |u|^p \\ &= \frac{r_1^3}{r_2^3} Q(r_2^{1/2} u^{t_1 \frac{r_2}{r_1}}) + \frac{3(p-2)}{2p}\lambda \Big(\frac{r_2}{r_1}\Big)^{p-3}\Big[\Big(\frac{r_2}{r_1}\Big)^{2(p-3)} - 1\Big] \int |r_1^{1/2} u^{t_1}|^p. \end{aligned}$$

Since $Q(r_1^{1/2}u^{t_1}) = 0$, $r_1 < r_2$ and $p > 3$, we conclude that

$$Q(r_2^{1/2} u^{t_1 \frac{r_2}{r_1}}) < 0,$$

and hence from Proposition 4.5 item (III)-(1), it follows that $t_r^-(u) < t_1 \frac{r_2}{r_1} < t_2$. Therefore

$$\begin{aligned} E(r_1^{1/2}u^{t_1}) &= \frac{r_1^3}{r_2^3} E(r_2^{1/2} u^{t_1 \frac{r_2}{r_1}}) + \frac{\lambda}{p}\Big(\frac{r_2}{r_1}\Big)^{p-3}\Big[\Big(\frac{r_2}{r_1}\Big)^{2(p-3)} - 1\Big] \int |r_1^{1/2} u^{t_1}|^p, \\ &> \frac{r_1^3}{r_2^3} E(r_2^{1/2} u^{t_2}) + \frac{\lambda}{p}\Big(\frac{r_2}{r_1}\Big)^{p-3}\Big[\Big(\frac{r_2}{r_1}\Big)^{2(p-3)} - 1\Big] \int |r_1^{1/2} u^{t_1}|^p. \end{aligned} \tag{A.1}$$

Since $r_1^{1/2}u^{t_1} \in \mathcal{N}_{r_1}$, it follows from Proposition 4.3 that there exists a constant $c_p' > 0$ such that

$$\int |r_1^{1/2} u^{t_1}|^p \geq \frac{c_p'}{\lambda r_1}, \quad \forall u \in \mathcal{M}_{r_1},$$

and consequently from (A.1) we conclude that

$$E(r_1^{1/2}u^{t_1}) \geq \frac{r_1^3}{r_2^3} E(r_2^{1/2} u^{t_2}) + \frac{c_p'}{r_1}\Big(\frac{r_2}{r_1}\Big)^{p-3}\Big[\Big(\frac{r_2}{r_1}\Big)^{2(p-3)} - 1\Big], \quad \forall u \in \mathcal{M}_{r_1}.$$

Therefore

$$I_{r_2} < \Big(\frac{r_2}{r_1}\Big)^3 I_{r_1} - \frac{c_p'}{r_1}\Big(\frac{r_2}{r_1}\Big)^p\Big[\Big(\frac{r_2}{r_1}\Big)^{2(p-3)} - 1\Big]$$

which concludes the proof. ∎

ACKNOWLEDGMENT. The authors would like to thank the anonimous referee for the attention paid to the paper and the useful suggestions which improved the reading of the manuscript.

References

[1] J. Bellazzini, L. Jeanjean and T. Luo, *Existence and instability of standing waves with prescribed norm for a class of Schrödinger–Poisson equations*, Proceedings of the London Mathematical Society **107** (2013), 303–339.

[2] J. Bellazzini and G. Siciliano, *Stable standing waves for a class of nonlinear Schrödinger–Poisson equations*, Zeitschrift für Angewandte Mathematik und Physik **62** (2010), 267–280.

[3] J. Bellazzini and G. Siciliano, *Scaling properties of functionals and existence of constrained minimizers*, Journal of Functional Analysis **261** (2011), 2486–2507.

[4] V. Benci, *Hylomorphic solitons*, Milan Journal of Mathematics **77** (2009), 271 – 332.

[5] V. Benci and D. Fortunato, *An eigenvalue problem for the Schrödinger–Maxwell equations*, Topological Methods in Nonlinear Analysis **11** (1998), 283–293.

[6] O. Bokanowski and N. Mauser, *Local approximation for the Hartree–Fock exchange potential: a deformation approach*, Mathematical Models and Methods in Applied Sciences **9** (1999), 941–961.

[7] I. Catto, J. Dolbeault, O. Sánchez and J. Soler, *Existence of steady states for the Maxwell–Schrödinger–Poisson system: exploring the applicability of the concentration-compactness principle*, Mathematical Models and Methods in Applied Sciences **23** (2013), 1915–1938.

[8] I. Catto and P. L. Lions, *Binding of atoms and stability of molecules in Hartree and Thomas–Fermi type theories. I. A necessary and sufficient condition for the stability of general molecular systems*, Communications in Partial Differential Equations **17** (1992), 1051–1110.

[9] M. Colin and T. Watanabe, *Standing waves for the nonlinear Schrödinger equation coupled with the Maxwell equation*, Nonlinearity **30** (2017), 1920–1947.

[10] Y. Il'yasov *On extreme values of Nehari manifold method via nonlinear Rayleigh's quotient*, Topological Methods in Nonlinear Analysis **49** (2017), 683–714.

[11] Y. Il'yasov and K. Silva, *On branches of positive solutions for $p-$Laplacian problems at the extreme value of the Nehari manifold method*, Proceedings of the American Mathematical Society **146** (2018), 2925–2935.

[12] L. Jeanjean and T. Luo, *Sharp nonexistence results of prescribed L^2-norm solutions for some class of Schrödinger–Poisson and quasi-linear equations*, Zeitschrift für Angewandte Mathematik und Physik **64** (2013), 937–954.

[13] L. Jeanjean, T. Luo and Z.-Q. Wang, *Multiple normalized solutions for quasi-linear Schrödinger equations*, Journal of Differential Equations **259** (2015), 3894–3928.

[14] E. H. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics, Vol. 14, American Mathematical Society, Providence, RI, 2001

[15] E. H. Lieb and B. Simon, *The Hartree–Fock theory for Coulomb systems*, Communications in Mathematical Physics **53**, (1977) 185–194.

[16] P. L. Lions, *The concentration-compactness principle in the calculus of variation. The locally compact case. I, II*, Annales de l'Institut Henri Poincaré. Analyse Non Linéaire **1** (1984), 109–145, 223–283.

[17] P. L. Lions, *Solutions of Hartree–Fock equations for Coulomb systems,* Communications in Mathematical Physics **109** (1987), 33–97.

[18] N. J. Mauser, *The Schrödinger–Poisson-$X\alpha$ equation*, Applied Mathematics Letters **14** (2001) 759–763.

[19] R. G. Parr and W. Yang, *Density Functional Theory of Atoms and Molecules*, Oxford University Press, New York, 1989.

[20] S. I. Pohozaev, *The fibration method for solving nonlinear boundary value problems*, Trudy Matematicheskogo Instituta imeni V. A. Steklova **192** (1990), 146–163. English translation: *Differential Equations and Function Spaces*, Proceedings of the Steklov Institute of Mathematics, Vol. 192, American Mathematical Society, Providence, RI, 1992, pp. 157–163.

[21] D. Ruiz, *The Schrödinger–Poisson equation under the effect of a nonlinear local term*, Journal of Funcional Analysis **237** (2006), 655–674.

[22] O. Sánchez and J. Soler, *Long-time dynamics of the Schrödinger–Poisson-Slater system*, Journal of Statistical Physics **114** (2004), 179–204.

[23] G. Siciliano and K. Silva, *The fibering method approach for a non-linear Schrödinger equation coupled with the electromagnetic field*, Publicacions Matemàtiques **64** (2020), 373–390.

[24] K. Silva, *On an abstract bifurcation result concerning homogeneous potential operators with applications to PDEs*, Journal of Differential Equations **269** (2020), 7643–7675.

[25] J. C. Slater, *A simplification of the Hartree-Fock method*, Physics Review **81** (1951) 385–390.

[26] L. Zhao and F. Zhao, *On the existence of solutions for the Schrödinger–Poisson equations*, Journal of Mathematical Analysis and Applications **346** (2008), 155–169.